Toric Graph Ideals

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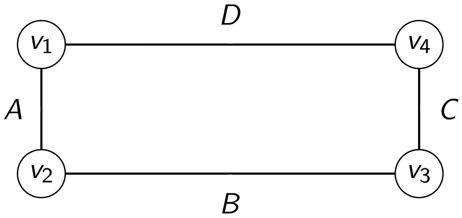
k is a field. Let G denote the following graph.

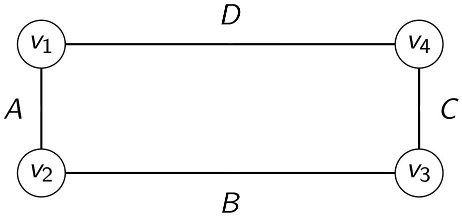


Define the homomorphism $\phi_G : k[A, B, C, D] \to k[v_1, v_2, v_3, v_4]$ by

$$A \mapsto v_1 v_2 \qquad B \mapsto v_2 v_3 \qquad C \mapsto v_3 v_4 \qquad D \mapsto v_4 v_1.$$

$$Im(\phi_G) = k[v_1 v_2, v_2 v_3, v_3 v_4, v_4 v_1].$$

Introduction (cont.)



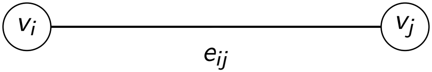
$$\phi_G(AC - BD) = 0$$

- In fact, it is easy to show that $ker(\phi_G) = \langle AC - BD \rangle$.
- We call this ideal, $ker(\phi_G)$, the Toric Ideal of the graph G.

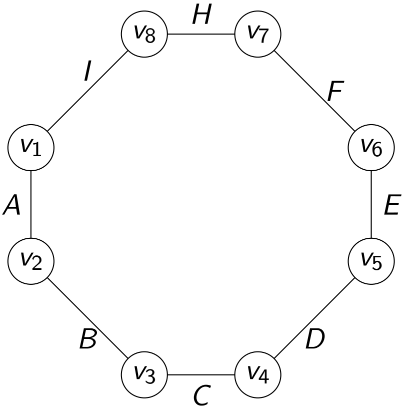
Graph Ideals

Definition

Let $G = (V, E)$ be an undirected graph. We can define the map $\phi_G : k[E] \to k[V]$ by $e_{ij} \mapsto v_i v_j$ as shown below. The ideal $\ker(\phi_G)$ is called the **Toric Ideal** *associated to G and is denoted I_G.*

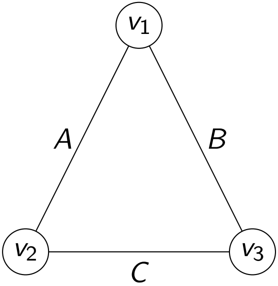


An Even Cycle



$$\phi_G(ACEH - BDFI) = v_1v_2v_3v_4v_5v_6v_7v_8 - v_2v_3v_4v_5v_6v_7v_8v_1 = 0$$

An Odd Cycle



Let's try to find a non-zero element in the toric ideal

$$AB\cdots - C\cdots$$

For this graph, the toric ideal is...
is..
hm..
trivial? What happened here?

Things We Want To Be True That Are!

Proposition

I_G is a homogeneous ideal generated by the following binomials which correspond to closed even walks of the graph:

$$B_w = \prod_{j=1}^{k} e_{2j-1} - \prod_{j=1}^{k} e_{2j}$$

Closed even walk:

$$w : w_1 \xrightarrow{e_1} w_2 \xrightarrow{e_2} \dots \xrightarrow{e_{2k-2}} w_{2k-1} \xrightarrow{e_{2k-1}} w_{2k} \xrightarrow{e_{2k}} w_1$$

$$w_i \in V, e_i \in E$$

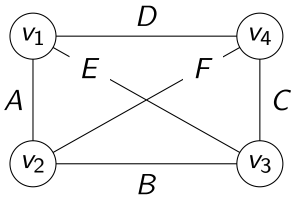
Minimal Generating Set

Definition

If $\langle s_1, ..., s_n \rangle = I$, then we say it is a **minimal generating set** if no subset of it generates I.

Notation:
$\mathcal{M}_G$ - any minimal generating set

An Example: The Complete Graph



$$\begin{array}{l} \quad AC - BD \\ -(AC - EF) \\ \hline EF - BD \end{array}$$

- $\mid \mathcal{M}_G \mid = 2$
- $\langle AC - BD, AC - EF, BD - EF \rangle$ seems like a more complete description of the toric ideal.
- The set $\{AC - BD, AC - EF, BD - EF\}$ forms what is called a **Graver Basis** for the ideal I_G.

Graver Bases

Notation: $\mathcal{G}_G$ - Graver Basis

Definition

- A binomial $x^{v^+} - x^{v^-} \in I_G$ is called **primitive** if $\nexists$ another binomial $x^{u^+} - x^{u^-} \in I_G$ s.t. $x^{u^+} | x^{v^+}$ and $x^{u^-} | x^{v^-}$.
- The set of all primitive binomials in I_G is called its **Graver basis** and is denoted $\mathcal{G}_G$.

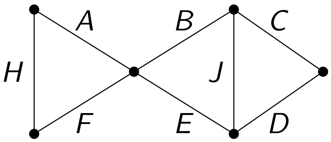
Fact!

Fact: $\mathcal{M}_G \subset \mathcal{G}_G$

BIG QUESTION:

- For what graphs G is $\mathrm{M}_G = \mathcal{G}_G$?

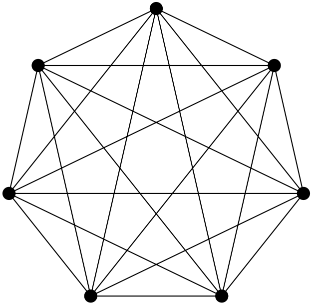
An Example of When $\mathcal{M}_G \neq \mathcal{G}_G$



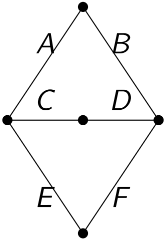
$\mathcal{M}_G = \{AJF - BEH, BD - CE\}$
$\mathcal{G}_G = \{AJF - BEH, BD - CE, CAJF - DB^2H, DAJF - CE^2H\}$

Note: $CAJF - DB^2H = BH(CE - BD) + C(AJF - BEH)$

$\mathcal{M}_G$ and $\mathcal{G}_G$ Can Be Vastly Different Sizes



$\mathcal{M}_G = 70$, but $\mathcal{G}_G = 3360$



(Minimal) closed even walks are all cycles, corresponding to

$$p_1 = AD - BC$$

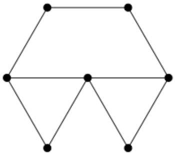
$$p_2 = CF - DE$$

$$p_3 = AF - BE$$

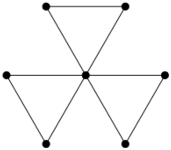
but $p_i \notin \langle p_j, p_k \rangle$

for $\{i, j, k\} = \{1, 2, 3\}$

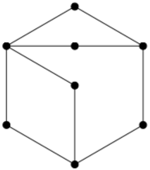
A Few More Examples Where $\mathcal{M}_G = \mathcal{G}_G$



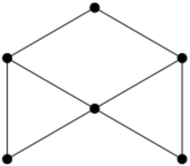
$|\mathcal{G}_G| = 3$



$|\mathcal{G}_G| = 3$

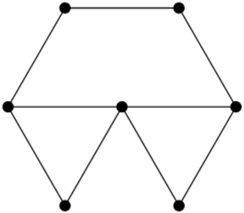


$|\mathcal{G}_G| = 6$

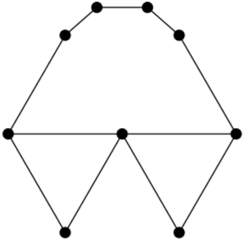


$|\mathcal{G}_G| = 3$

Contructing Larger Graphs Where $\mathcal{M}_G = \mathcal{G}_G$ From Smaller Ones



$\mapsto$



Universal Gröbner Basis

- Another kind of basis with some algorithmically nice properities is called a **Universal Gröbner Basis**, denoted $\mathcal{U}_G$, and this set lays nestled in between the minimal generating set of a toric ideal and the Graver Basis

Proposition

$\mathcal{M}_G \subset \mathcal{U}_G \subset \mathcal{G}_G$ *for any minimal generating set* $\mathcal{M}_G$

Some Results

Theorem

$\mathcal{M}_G = \mathcal{U}_G \Leftrightarrow \mathcal{M}_G = \mathcal{G}_G$. *If* I_G *sastisfies either of these conditions we call it robust.*

Theorem

I_G *is robust* $\Leftrightarrow$ *(Graph theoretic conditions).*

Some Results

Theorem

$\mathcal{M}_G = \mathcal{U}_G \Leftrightarrow \mathcal{M}_G = \mathcal{G}_G$. If I_G sastisfies either of these conditions we call it robust.

Theorem

I_G is robust $\Leftrightarrow$ (Graph theoretic conditions).

Thank You!

References

Sturmfels, Bernd. *Gröbner Bases and Convex Polytopes.* American Mathematical Society: Providence, RI, 1991.