Ann. Comb. 19 (2015) 641–660
DOI 10.1007/s00026-015-0288-3
Published online October 6, 2015

Annals of Combinatorics

Robust Graph Ideals

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Received November 26, 2013

Mathematics Subject Classification: 13P10, 05C25

Abstract. Let I be a toric ideal. We say I is *robust* if its universal Gröbner basis is a minimal generating set. We show that any robust toric ideal arising from a graph G is also minimally generated by its Graver basis. We then completely characterize all graphs which give rise to robust ideals. Our characterization shows that robustness can be determined solely in terms of graph-theoretic conditions on the set of circuits of G.

Keywords: toric ideals, universal Gröbner bases, graph ideals

1. Introduction

Let $A = (\mathbf{a}_1|\mathbf{a}_2|\cdots|\mathbf{a}_m)$ be an $n \times m$ matrix with entries in $\mathbb{N}$. Consider the homomorphism $\phi\colon k[x_1, \ldots, x_m] \to k[s_1, \ldots, s_n]$ such that $x_i \mapsto \mathbf{s}^{\mathbf{a}_i}$, where by convention $\mathbf{s}^{\mathbf{v}} := s_1^{v_1} \cdots s_n^{v_n}$ for $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{N}^n$. The *toric ideal* I_A is defined to be $\ker(\phi)$. Toric ideals arise naturally in several areas of study, including integer programming, algebraic statistics, geometric modeling, and graph theory (see [1, 3, 5]).

It is well-known that toric ideals are prime ideals that are generated by binomials [6, §4]. Among these, distinguished sets of binomials in I_A have been introduced and studied for many classes of toric ideals. The *Graver basis* of I_A, denoted $\mathcal{G}_A$, consists of all binomials which are *primitive*; that is, all nonzero binomials $\mathbf{x}^{\mathbf{c}} - \mathbf{x}^{\mathbf{d}} \in I_A$ such that there does not exist a binomial $\mathbf{x}^{\mathbf{c}'} - \mathbf{x}^{\mathbf{d}'} \in I_A$ with $\mathbf{x}^{\mathbf{c}'} \mid \mathbf{x}^{\mathbf{c}}$ and $\mathbf{x}^{\mathbf{d}'} \mid \mathbf{x}^{\mathbf{d}}$. The *universal Gröbner basis* of I_A, denoted $\mathcal{U}_A$, is the union of all reduced Gröbner bases for I_A. Furthermore, $\mathcal{U}_A$ is a Gröbner basis for I_A under all monomial term orders. Finally, we say that $\mathbf{x}^{\mathbf{c}} - \mathbf{x}^{\mathbf{d}}$ is a *circuit* if it is irreducible and if the set of indices for which c_i, d_i are nonzero is minimal with respect to inclusion. Let $\mathcal{C}_A$ be the set of all circuits in I_A. By a result of [6], the inclusions $\mathcal{C}_A \subset \mathcal{U}_A \subset \mathcal{G}_A$ hold. Typically these inclusions are strict.

In this paper, we study toric ideals for which $\mathcal{U}_A$ is a minimal generating set for I_A. We call these ideals *robust*. In [2], the authors classified all robust toric ideals generated by quadratics; however, there are significant obstacles in characterizing robustness for toric ideals generated in higher degrees. The purpose of this project is to characterize robustness for toric ideals arising from graphs, that is, when A is the incidence matrix of a graph. Our first main result shows that for graphs, robustness is strong enough to ensure that the universal Gröbner basis and Graver basis are equal:

Theorem 1.1. *Let G be a simple graph. Then I_G is robust iff it is minimally generated by its Graver basis.*

This result is quite surprising, as it states that minimality of $\mathcal{U}_G$ implies that of $\mathcal{G}_G$. This behavior was witnessed for some classes of hypergraph ideals in [3] as well. It is open whether or not this holds for general toric ideals. The proof of Theorem 1.1 relies on characterizations of $\mathcal{U}_G$ and $\mathcal{G}_G$ given in [5, 7]. We then use graph-theoretic analysis of primitive binomials to complete the proof.

Next, we characterize all graphs G that give rise to robust ideals. Given Theorem 1.1 this turns out to be equivalent to requiring that every primitive binomial is *indispensable*–that is, it is contained in every set of minimal generators of I_G. The following theorem is stated in terms of graph theoretic properties of the circuits of the graph G, using a graph theoretic characterization of circuits in [8].

Theorem 1.2. *I_G is robust if and only if the following conditions are satisfied.*

R1: *No circuit of G has an even chord,*
R2: *No circuit of G has a bridge,*
R3: *No circuit of G contains an effective crossing, and*
R4: *No circuit of G shares exactly one edge (and no other vertices) with another circuit such that the shared edge is part of a cyclic block in both circuits.*

The layout of the paper is as follows: In Section 2, we review the construction of toric graph ideals, definitions relating to their study, and characterizations of circuits, the Graver basis, and the universal Gröbner basis of such an ideal. In Section 3, we prove that a robust toric graph ideal is minimally generated by its Graver basis, which facilitates major results in Section 4, where we present a graph-theoretic characterization of such ideals. In Section 5, we apply our results to list toric graph ideals generated in low degrees. Finally, we conclude with some open questions in the setting of general toric ideals.

2. Toric Graph Ideals

Let G be a finite, simple graph with edge set $E(G)$ and vertex set $V(G)$. We define the *toric graph ideal* of G to be the toric ideal associated with the homomorphism $\phi_G\colon k[E(G)] \to k[V(G)]$ such that $\phi_G(e_{ij}) = v_i v_j$. Equivalently, ϕ_G sends an edge of G to the product of its corresponding vertices. We denote such an ideal by I_G.

A *walk* is a finite sequence of the form

$$w = (\{v_{i_1}, v_{i_2}\}, \{v_{i_2}, v_{i_3}\}, \dots, \{v_{i_{n-1}}, v_{i_n}\}),$$

with each $v_{i_j} \in V(G)$ and $e_{i_j} = \{v_{i_j}, v_{i_{j+1}}\} \in E(G)$. We denote a walk w either by its sequence of edges, $(e_1, \dots, e_k)$, or by its sequence of vertices, $(v_1, \dots, v_k)$. A *closed* walk is a walk with $v_{i_1} = v_{i_n}$. A walk $w = (e_{i_1}, e_{i_2}, \dots, e_{i_n})$ is called *even* (*odd*, respectively) if n is *even* (*odd*, respectively). A walk w' is called a *subwalk* of another walk w if the edges of w' appear in the same order in w.

Given a closed even walk, $w = (e_1, e_2, \dots, e_{2k})$, we denote its corresponding binomial

$$B_w = \prod_{i=1}^{k} e_{2i-1} - \prod_{i=1}^{k} e_{2i} \in I_G.$$

From [8], we know that the ideal I_G is generated by binomials of the above form; furthermore, every binomial generator arises through this correspondence [4, Lem. 1.1]. Given such a walk, let w^+ denote the set of edges with odd indices. Similarly, define w^- to be the set of edges with even indices. We define edges of odd index as *odd edges* and define *even edges* analogously. Two edges are said to have the same *parity* if they are both in w^+ or w^-. Put $E^+(w) = \prod_{i=1}^{k} e_{2i-1}$ and $E^-(w) = \prod_{i=1}^{k} e_{2i}$ so that $B_w = E^+(w) - E^-(w)$.

Many ideal-theoretic properties of I_G can be interpreted graph theoretically. To develop this relationship, we present some basic facts about robust toric ideals. The circuits of graph ideals have a combinatorial characterization. In the following, by the *graph of a walk* w *on a graph* G we mean the subgraph of G consisting of the vertices and edges that appear in w.

Definition 2.1. *A* simple path *of a graph G is a walk $(v_1, v_2, \dots, v_k)$ such that the v_i are all distinct.*

Proposition 2.2. ([8]) *B_w is a circuit of I_G iff the graph of w is one of the following*:

(C1) *an even cycle,*
(C2) *two odd cycles joined at a single vertex,*
(C3) *two vertex-disjoint odd cycles joined by a simple path $p = (v_1, v_2, \dots, v_k)$ with $k > 1$ such that the intersection of p with the first* (*second, respectively*) *cycle is the first* (*last, respectively*) *vertex of p.*

Circuits of the aformentioned types (and, by abuse of notation, the associated walks) will be referred to as C1, C2, and C3 circuits, respectively.

Similar conditions are needed for B_w to be primitive.

Proposition 2.3. ([4, Lem. 3.2]) *If B_w is primitive, then the graph of w necessarily is of one of the following forms*:

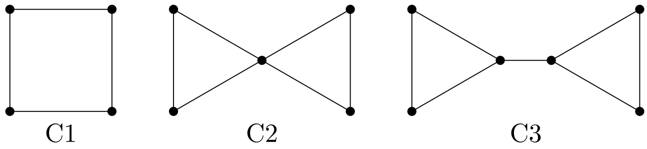


Figure 1: Examples of circuits of type C1, C2, and C3.

(P1) *an even cycle,*

(P2) *two odd cycles joined at a single vertex,*

(P3) (c_1, w_1, c_2, w_2) *where* c_1, c_2 *are vertex disjoint cycles and* w_1, w_2 *are walks which combine a vertex* v_1 *of* c_1 *and a vertex* v_2 *of* c_2.

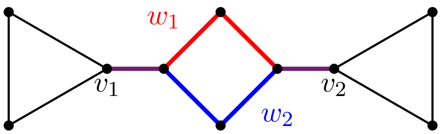


Figure 2: Example of a P3 primitive walk.

Primitive elements of the aforementioned types (and their associated walks) will be referred to as P1, P2, and P3 elements, respectively. Note that all P1 (P2, respectively) primitive elements are also C1 (C2, respectively) circuits, so primitive noncircuits must be of type P3.

However, there exist walks of the third type, P3, which give rise to non-primitive binomials. Necessary and sufficient conditions for primitivity require the introduction of new terminology. The definitions which follow are borrowed from [7].

A *cut vertex* (*cut edge*, respectively) of a graph is a vertex (edge, respectively) whose removal increases the number of connected components of a graph. A graph is *biconnected* if it is connected and does not contain a cut vertex. A *block* of a graph G is a maximal biconnected subgraph of G. A *cyclic block* is a block which is 2-regular; namely, each vertex in the block is contained in exactly two of its edges. A *sink* of a block B with respect to a walk w is a common vertex of two edges in B which have the same parity in w. By abuse of terminology, we will often refer simply to a sink of B when B is understood to be contained in the graph of w.

While G does not have multiple edges, we say an edge e is a *multiple edge of a walk* if e appears more than once in the walk. We define a walk to be *strongly primitive* if it is primitive and does not contain two sinks within distance 1 of each other in any cyclic block. These notions yield a graph-theoretic description of the Graver basis $\mathcal{G}_G$.

Theorem 2.4. ([5]) B_w *is primitive iff the following hold*:

(1) *every block in the graph of* w *is a cycle or a cut edge,*

(2) *every multiple edge of w is a cut edge in the graph of w and is traversed exactly twice,*
(3) *every cut vertex in the graph of w belongs to exactly 2 blocks of the graph of w and is a sink of both.*

Still more care is required to describe $\mathcal{U}_G$ and the minimal generators. A cyclic block of a primitive walk is *pure* if all of its edges have the same parity. We have the following result from [7]:

Theorem 2.5. *Let w be a primitive walk. Then $B_w \in \mathcal{U}_G$ iff no cyclic block of the graph of w is pure.*

In view of the containments $\mathcal{C}_G \subset \mathcal{U}_G \subset \mathcal{G}_G$ and the first two propositions, circuits of types C1 and C2 are always in $\mathcal{U}_G$. Then, by using Theorems 2.4 and 2.5, we have the following result.

Corollary 2.6. *Let $B_w \in \mathcal{U}_G$. Then, the walk w satisfies one of the following*:

(A) *the graph of w is an even cycle,*
(B) *the graph of w is two odd cycles joined at a single vertex,*
(C) *w is a walk of the form (c_1, w_1, c_2, w_2), where c_1, c_2 are vertex disjoint and w_1, w_2 are walks connecting them, subject to the conditions on w*:

 i) *every block of the graph of w is a cycle or a cut edge,*
 ii) *every multiple edge of w is a double edge and a cut edge of the graph of w,*
 iii) *every cut vertex of the graph of w belongs to exactly 2 blocks and is a sink of both,*
 iv) *no cyclic block of the graph of w is pure.*

Finally, to study the minimal generators of a toric graph ideal, we must understand how the closed even walks relate to the larger graph. An edge $f \in E(G)$ is said to be a *chord* of a walk w if both of its vertices belong to w but f itself does not. Chords fall into three classes. A *bridge* $f = \{v_1, v_2\}$ of a primitive walk $w = (e_1, e_2, \ldots, e_{2k})$ is a chord such that w contains two different blocks B_1, B_2 with $v_1 \in B_1$ and $v_2 \in B_2$. A chord $f = \{v_i, v_j\}$ that is not a bridge is called *even* (*odd*, respectively) if the walks $(e_1, e_2, \ldots, e_{i-1}, f, e_j, e_{j+1}, \ldots, e_{2k})$ and $(e_i, e_{i+1}, \ldots, e_{j-1}, f)$ are both even (odd, respectively). Note that a chord starting at a cut vertex is always a bridge, since it is contained in two distinct blocks. Due to a result in [5], binomials that occur in a minimal generating set arise from walks that are necessarily strongly primitive and contain no even chords or bridges (see Section 4).

Let $w = ((v_1, v_2), (v_2, v_3), \ldots, (v_{2n}, v_1))$ be a primitive walk. Let $f = (v_i, v_j)$ and $f' = (v_k, v_\ell)$ be two odd chords such that $j - i, \ell - k \in 2\mathbb{N}$, with $1 \leq i < j \leq 2n$ and $1 \leq k < l \leq 2n$. Then, f and f' *cross effectively* if $i - k$ is odd and either $i < k < j < \ell$ or $k < i < \ell < j$. Note that if two odd chords f and f' cross effectively in w, then all their vertices are in the same cyclic block of w. We say *w has an effective crossing* if two odd chords f, f' of w exist that cross effectively.

From here, if w is a walk of G, we say w^{-1} to denote w traversed in the opposite direction. So, if $w = (e_1, \ldots, e_n)$, then $w^{-1} = (e_n, \ldots, e_1)$.

3. Graver Bases and Robustness

We say that the toric graph ideal I_G is *robust* if $\mathcal{U}_G$ is a minimal generating set for I_G. We call a graph G robust if I_G is robust. Robustness is a relatively strong property as it ensures, for instance, that all initial ideals have the same minimal number of generators:

$$\mu(I_G) = \mu(\mathrm{in}_< I_G) \text{ for all term orders } <.$$

In terms of the binomials themselves we will use the following necessary condition:

Lemma 3.1. *If I_G is robust, then no term of an element of $\mathcal{U}_G$ can divide a term of another element of $\mathcal{U}_G$.*

Proof. Suppose $\mathcal{U}_A$ contains binomials

$$f = m_1 - m_2, \qquad g = n_1 - n_2$$

with m_1 dividing n_1. Then some variable x divides m_1 but not m_2 by primality of I_G. Taking $<$ to be the Lex term order with x first, we see that $(\mathrm{in}_< f) \mid (\mathrm{in}_< g)$. Thus, $\mu(\mathrm{in}_< I_G)) < |\mathcal{U}_G| = \mu(I_G)$, a contradiction. ∎

Our first main result states that the containment $\mathcal{U}_G \subset \mathcal{G}_G$ is an equality if G is a robust graph.

Theorem 3.2. *I_G is robust iff it is minimally generated by its Graver basis.*

Proof. If I_G is minimally generated by $\mathcal{G}_G$ then since $\mathcal{U}_G \subset \mathcal{G}_G$, and both generate I_G, it follows that $\mathcal{U}_G$ is also a minimal generating set. Hence, I_G is robust.

To prove the other direction we will prove the contrapositive. We assume that there is a primitive walk w of G whose corresponding binomial $b_w = E^+(w) - E^-(w)$ is not in $\mathcal{U}_G$, then we construct another primitive walk w' whose binomial is in $\mathcal{U}_G$ but is not minimal, so that G is not robust.

Let such a w as above be given. Since $b_w \notin \mathcal{U}_G$, by Theorem 2.5, the graph of w must contain at least one pure cyclic block B. First, we want to show that we can assume, without loss of generality, that the graph of w contains exactly one pure cyclic block.

Suppose that w is primitive and that its graph contains more than one pure cyclic block. There can only be finitely many blocks since G is finite; pick one and call it B. Thus, B can be written as $(e_1, e_2, \ldots, e_n)$, where we assume that all of the e_j belong to w^-. Then w must be of the form $(w_1, e_1, w_2, e_2, \ldots, w_n, e_n)$, where each w_j is a subwalk of w that starts and ends at vertex j, as in Figure 3. Further, since w is primitive and each of the edges e_i is odd, the subwalks w_j must have odd length.

Now, let w' be the walk $w' = (w_1, e_1, \ldots, w_{n-2}, e_{n-2}, e_{n-1}, e_n)$ that follows the same path as w, only skipping over the last two odd walks w_{n-1} and w_n. Then, B is still a cyclic block of the graph of w', but it is not a pure cyclic block since the edge e_{n-1} belongs to w^+ instead of w^-. This also means that the graph of w' has strictly fewer pure cyclic blocks than the graph of w does. It is possible that the graph of w' now has no pure cyclic blocks, if all of the other pure cyclic blocks of the graph of w were contained in the graphs of w_{n-1} or w_n. In this case, we could have chosen w' to

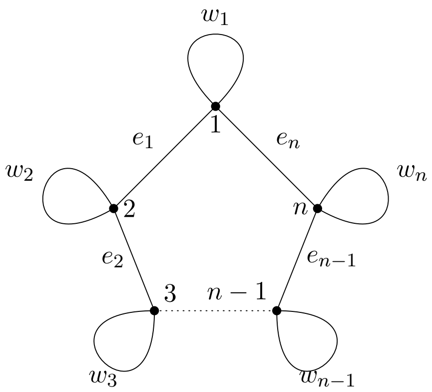


Figure 3: A primitive walk w with cyclic block $(e_1, e_2, \ldots, e_n)$ where $n \geq 4$.

omit the walks w_1 and w_2, so that at least some of the remaining pure cyclic blocks of the graph of w are not contained in the graphs of w_1 and w_2. Therefore, the graph of w' has at least one pure cyclic block, since we assumed that the graph of w had more than one pure cyclic block.

We check that w' is still primitive in this construction, using Theorem 2.4. First, suppose that the graph of w' has a block B' that is not a cycle or a cut edge. Then, since the graph of w' is a subgraph of the graph of w, B' must be contained in a block that lies in the graph of w. But then this block can't be a cycle or a cut edge, contradicting the primitiveness of w. Now let e be a multiple edge of w', so that it is also a multiple edge of w. If it were traversed more than twice in w', then it would be traversed more than twice in w, again contradicting the primitiveness of w. Similarly, if e is not a cut edge of the graph of w', then it is not a cut edge of the graph of w, again since the graph of w' is a subgraph of the graph of w.

Finally, let v be a cut vertex in the graph of w'. First, suppose that v is contained in the block B, that is, it is one of the vertices $1, 2, \ldots, n-2$. In this case, it is clear that v is also a cut vertex of the graph of w, so it belongs to exactly two blocks: B and a block B' of the graph of some w_j, and it is a sink of both. Since B and B' are also blocks of the graph of w' and the parities of all the edges of w' are the same as their parities in the walk w, with the exception of the edge e_{n-1}, we have that v must satisfy the same property when considered as a cut vertex of w'. Now suppose that v is contained in one of the w_j, $1 \leq j \leq n-2$. Without loss of generality, say that v is contained in w_1. Then, since v is a cut vertex of the graph of w', it must also be a cut vertex of the graph of w_1, since w_1 is disjoint from the other w_j. Thus, it is also a cut vertex of the graph of w. As such, it belongs to exactly two blocks B', B'' of the graph of w. But these blocks are contained in the graph of w_1, so they must also be blocks in the graph of w', so that v belongs to exactly two blocks in the graph of w'. Similarly, since v is a sink of both of them when considered as a vertex of w, this property also holds when we consider v as a vertex of w' by the parity argument above. Thus, by Theorem 2.4, the walk w' is primitive.

If the graph of w' has more than one pure cyclic block, we can repeat this construction on another pure cyclic block of the graph of w' to get another primitive subwalk that has strictly fewer pure blocks than the graph of w' has, but that has at least one. We can repeat this process until it terminates at a walk with exactly one pure cyclic block. Call this walk w.

Since the graph of w has one pure cyclic block B, the binomial b_w is not contained in $\mathcal{U}_G$ but is primitive. Suppose that B has at least 4 edges, so $n \geq 4$. Now, repeat the construction above to get a subwalk w' of w whose graph has no pure cyclic blocks. By Theorem 2.5, this means that $b_{w'}$ is contained in the universal Gröbner basis of I_G. However, the edges $e_1, e_2, \dots, e_{n-2}, e_n$ are all contained in w^-, which means that the vertices $1, 2, \dots, n-2$ are all sinks of the block B of the graph of w'. Since $n \geq 4$, vertices 1 and 2 are both sinks, and they have distance one since they are connected by the edge e_1, so that w' is not strongly primitive. By the result of [5] discussed above, this implies that w' is not minimal. Since we have an element of $\mathcal{U}_G$ that is not minimal, thus not contained in a minimal set of generators, it must be the case that $\mathcal{U}_G$ is not a minimal generating set, so G is not robust.

Finally, we consider the special case where $n = 3$. That is, the single pure cyclic block B of w has only three edges e_1, e_2, e_3, as in Figure 4.

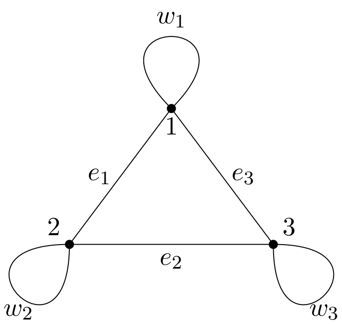


Figure 4: A primitive walk w with cyclic block $(e_1, e_2, \dots, e_n)$ where $n = 3$.

Now let w' be the primitive walk obtained by the construction above, that is, $w' = (w_1, e_1, e_2, e_3)$. Let w'' be the closed even walk $w'' = (w_1, e_1, e_2, w_3, e_2, e_1)$. Neither the graph of w' nor the graph of w'' have any pure cyclic blocks, since B was the only pure cyclic block of the graph of w by assumption, so as long as they are primitive, their corresponding binomials will be elements of $\mathcal{U}_G$. By the above construction, w' is primitive, and it is easy to see that w'' is as well, using the fact that w is primitive. Then, the binomial corresponding to the walk w'' is $b_{w''} = w_1^+ e_2^2 w_3^- - w_1^- e_1^2 w_3^+$, where w_j^+ is the odd part of w_j, and w_j^- is the even part of w_j. Similarly, the binomial corresponding to w' is $b_{w'} = w_1^+ e_2 - w_1^- e_1 e_3$. By the above argument, both of these are elements of $\mathcal{U}_G$. However, one term of $b_{w'}$ divides a term of $b_{w''}$. By Lemma 3.1, this implies that $\mathcal{U}_G$ is not robust. ∎

The referees have pointed out that the method of the proof of Proposition 4.6 provides another proof. Since it is short and of a different flavor we include it here:

Alternate Proof. Suppose as before that there is a primitive walk w of G such that the graph of w contains a pure cyclic block B. Let e be an edge in B connecting two sinks s_1, s_2 in B. Applying Lemma 4.5 to s_1 on the part of the graph not containing B we get from Lemma 4.4 an odd cycle c_1 and a simple path p_1 connecting c_1 to s_1. By the same argument applied to s_2, we get an odd cycle c_2 and a simple path p_2 connecting c_2 to s_2. Now let q be the path in B connecting s_1, s_2 that does not contain e. From the C3 circuit $c := \left(c_1, p_1, q, p_2^{-1}, c_2, p_2, q^{-1}, p_1^{-1}\right)$ we have that $B_c \in \mathcal{U}_G$. But c is not a minimal generator by the result of [5] mentioned before since e is a bridge of c, a contradiction. ∎

The following proposition gives an application of Theorem 3.2. It describes one modification to any graph G that preserves robustness. Example 3.4 shows that modifying nonrobust graphs can often have unpredictable effects on $\mathcal{U}_G$.

Proposition 3.3. *Let G be a graph and b be an edge. Let G' be the graph obtained from G by replacing b with three edges. Then $|\mathcal{G}_G| = |\mathcal{G}_{G'}|$. If G is robust, then so is G'.*

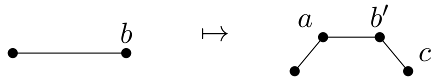


Figure 5: Construction in Proposition 3.3.

Proof. Consider when we replace the edge $b \in E(G)$ with $\{a, b', c\}$ as in Figure 5, producing a new graph G'. Notice that any primitive walk that contains one of a, b', c must contain them all. Let $\mathcal{L}_G$ denote the set of primitive walks on G. Define the map $\varphi\colon \mathcal{L}_G \to \mathcal{L}_{G'}$ which takes a walk w to its image in G replacing all instances of b with $\{a, b', c\}$. On binomials,

$$\varphi\left(mb^\ell - n\right) = m(ac)^\ell - n(b')^\ell,$$

where m, n are monomials not involving b. It is straightforward to check that φ provides a bijection between the primitive walks of G and G', proving that $|\mathcal{G}_G| = |\mathcal{G}_{G'}|$.

Suppose that G is robust. We will show that $\mathcal{G}_{G'}$ is a minimal generating set for $I_{G'}$. Let $\mathcal{G}_G = \{w_1, \ldots, w_k\}$ and $\mathcal{G}_{G'} = \{w'_1, \ldots, w'_k\}$ where $\varphi(w_j) = w'_j$. If the binomials $B_{w'_j}$ do not minimally generate $I_{G'}$ then one, say, $B_{w'_1}$ must be a polynomial linear combination of the others. But then this must mean that one term of $B_{w'_1}$ is divisible by a term of another $B_{w'_i}$, say $B_{w'_2}$. But then it follows that one term of B_{w_1} is divisible by a term of B_{w_2}. But by Theorem 3.2 we have that $\mathcal{U}_G = \mathcal{G}_G$ and this is a contradiction by Lemma 3.1. ∎

Example 3.4. Notice that for non-robust graphs, the number of minimal generators and the set $\mathcal{U}_G$ are very sensitive to changes in the graph G. For example, consider the graphs G and G' in Figure 6. These graphs are not robust. The left graph has

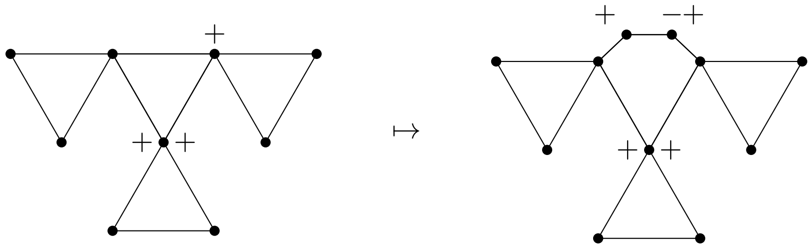


Figure 6: Example where $\mu(I_G) < \mu(I_{G'})$.

$\mu(I_G) = 6$, and the other satisfies $\mu(I_{G'}) = 7$. The walks w and w' that traverse each edge in G and G' once are both primitive, but $B_w \notin \mathcal{U}_G$ whereas $B_{w'} \in \mathcal{U}_{G'}$.

Example 3.5. The reverse implication in the proposition is false as shown in Figure 7. The graph on the right is obtained by contracting three edges into one. The graph on the left is robust, but the one on the right is not.

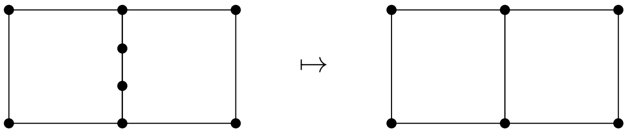


Figure 7: Counterexample to the reverse implication.

4. Characterization of Robust Graph Ideals

We begin this section with a definition and characterization of indispensable walks:

Definition 4.1. *A primitive walk w of a graph G is* indispensable *if the corresponding binomial B_w or its negation appears in every minimal generating set of I_G.*

Proposition 4.2. ([5, Thm. 4.14]) *A primitive walk w is indispensable if and only if*

I1: *w has no even chords,*
I2: *w has no bridges,*
I3: *w has no effective crossings, and*
I4: *w is strongly primitive.*

Lemma 4.3. *I_G is robust if and only if all primitive elements are indispensable.*

Proof. Since $\mathcal{U}$ is a generating set, we have the inclusions

$$\mathcal{I} \subset \mathcal{U} \subset \mathcal{G},$$

where $\mathcal{I}$ denotes the indispensable elements. Now if I_G is robust, then by Theorem 3.2, we have equalities everywhere, since in particular our minimal generating set is unique by the uniqueness of $\mathcal{U}$ and so $\mathcal{I} = \mathcal{U}$. Conversely, if $\mathcal{G} = \mathcal{I}, \mathcal{U} = \mathcal{I}$ and so we have robustness. ∎

Lemma 4.4. ([5, Cor. 3.3]) *A cut vertex in the graph W of a primitive walk w separates the graph in two vertex-disjoint parts, the total number of edges of the cyclic blocks in each part is odd.*

Lemma 4.5. *Suppose v_0 is a cut vertex in the graph W of a primitive walk w. Then, there exist two simple paths p_1, p_2 from v_0 to two odd cyclic blocks B_1, B_2 of W respectively, where B_1, B_2 are in the two different parts of W in Lemma* 4.4.

Proof. Let w be a primitive walk with cut vertex v_0. By Lemma 4.4, v_0 separates W into two connected subgraphs W_1, W_2 both of which have at least one odd cyclic block. Further, W_1, W_2 are vertex disjoint except for v_0. Now since the W_i are connected, we can find paths joining v_0 to odd cycles in each W_i. These can be chosen to be simple by omitting any "loops". ∎

To determine questions about robustness, we can simplify the characterization of indispensable elements given above. To make this clear, we offer the following proposition.

Proposition 4.6. *I_G is robust if and only if all primitive elements satisfy conditions I1, I2, and I3 of Proposition* 4.2.

Proof. If I_G is robust, then all primitive elements are indispensable by Lemma 4.3, so they satisfy conditions I1 through I4. In particular, they satisfy the first three. Now suppose that all primitive elements of I_G satisfy conditions I1, I2, and I3, but suppose for the sake of a contradiction that there exists a primitive walk w that does not satisfy I4, that is, it is not strongly primitive.

The walk w must be of type P1, P2, or P3.

P1: An even cycle is biconnected and its underlying graph is 2-regular thus no sinks exist so w is strongly primitive.

P2: For two odd cycles joined at a cut vertex, the cut vertex is the unique sink, so w is strongly primitive.

P3: Suppose w consists of two odd cycles joined along paths. Let e denote an edge connecting two sinks s_1, s_2 in the same cyclic block B. Applying Lemma 4.5 to s_1 on the part of the graph not containing B from Lemma 4.4, we get an odd cycle c_1 and a simple path p_1 connecting c_1 to s_1. By the same argument, we get an odd cycle c_2 and a simple path p_2 connecting c_2 to s_2. Now let q denote the path in B connecting s_1 and s_2 that does not contain e. Then, e forms a bridge of the C3 circuit $c := \left(c_1, p_1, q, p_2^{-1}, c_2, p_2, q^{-1}, p_1^{-1}\right)$, contradicting I2. ∎

We are now ready to state our classification of robust graph ideals.

Theorem 4.7. *I_G is robust if and only if the following conditions are satisfied.*

R1: *No circuit of G has an even chord,*
R2: *No circuit of G has a bridge,*
R3: *No circuit of G contains an effective crossing, and*
R4: *No circuit of G shares exactly one edge (and no other vertices) with another circuit such that the shared edge is part of a cyclic block in both circuits.*

In particular, this implies that questions of robustness can be answered by looking at the circuits as they lie on the graph.

Proof. By Proposition 4.6, it suffices to show that all primitive walks satisfy conditions I1, I2, and I3 if and only if G satisfies conditions R1 through R4. To do this, we'll show the contrapositive statement, which is that one of R1 through R4 is not satisfied if and only if there exists a primitive walk that doesn't satisfy one of I1 through I3. We begin with the forward direction of this new statement.

$\Rightarrow$: $\neg$ R1, R2, R3 Suppose that one of R1 through R3 is false. Then, there exists a circuit w with either an even chord, a bridge, or an effective crossing. But all circuits are primitive, so w is a primitive walk that doesn't satisfy one of I1 through I3.

$\neg$ R4 Suppose that R4 is false, then there are two circuits c and c' that share exactly one edge e (and no other vertices), where e belongs to a cyclic block of both c and c'. First consider the case where c and c' are C1 circuits. Then write $c = (e, w)$ where w is an odd simple walk connecting the vertices of e, and similarly put $c' = (e, w')$. Since c and c' share no vertices other than the two of e, the walks w and w' share only these two vertices as well. Consider the new walk $u = \left(w', w^{-1}\right)$ that starts and ends at one vertex of e. The walk u is an even cycle since w and w' are both odd only share the two vertices that connect them. However, e is an even chord of u, by construction. Therefore, u is primitive and doesn't satisfy I1.

Now consider the case where $c = (e, w)$ is a C1 circuit and $c' = \left(c'_1, w', c'_2, w'^{-1}\right)$ is a circuit of type C2 or C3. (In the case where it is C2, w' is the empty walk.) We can suppose, without loss of generality, that the edge e occurs in c'_2. Then, calling v' the vertex that connects c'_2 to w', let u'_1 be a subwalk of c'_2 connecting v' to one of the vertices of e, and u'_2 the subwalk of c'_2 connecting the other vertex of e to v' such that $c'_2 = (u'_1, e, u'_2)$. Consider the new walk $u = \left(c'_1, w', u'_1, w, u'_2, w'^{-1}\right)$. It is a C3 (or C2, if w' is an empty walk), by the disjointness condition on c and c'. If both u'_1 and u'_2 have positive length, then e is an even chord of u connecting two vertices in c'_2, neither of which is v', by construction. Thus, u doesn't satisfy condition I1. If one of u'_1, u'_2 is empty (they can't both be empty since c'_2 is not a loop), then e is a chord connecting v' to another vertex in c'_2, so that e is a bridge of u. In this case, u doesn't satisfy condition I2.

Finally, we can consider the case where $c = \left(c_1, w, c_2, w^{-1}\right)$ and $c' = \left(c'_1, w', c'_2, w'^{-1}\right)$ are both C2 or C3 circuits. Without loss of generality, suppose that e is contained in both c_2 and c'_2. Now let v be the vertex connecting c_2 to w, and define v' similarly for c'_2 and w'. As before, write $c_2 = (u_1, e, u_2)$ such that u_1 starts at v and u_2

ends at v. Now after possibly replacing c'_2 with $c_2'^{-1}$, let u'_2 be the subwalk of c'_2 starting at the last vertex of u_1 and ending at v', and let u'_1 be the subwalk of c'_2 starting at v' and ending at the first vertex of u_2; then, $c'_2 = (u'_1, e, u'_2)$. Now consider the closed even walk $u = \left(c_1, w, u_1, u'_2, w', c'_1, w'^{-1}, u'_1, u_2, w^{-1}\right)$. The walk u is primitive of type P3 since all blocks are either one of the cycles $c_1, c_2, (u_1, u'_2, u'_1, u_2)$ or one of the cut edges contained in w, w', and since the other properties in Theorem 2.4 are satisfied by construction of u. If none of u_1, u_2, u'_1, u'_2 are empty, then e is a chord of u connecting two vertices in the cyclic block (u_1, u'_2, u'_1, u_2), neither of which is v or v'. In this case, e is an even chord by construction, so that u doesn't satisfy I1. Alternatively, if one of these four walks is empty, then at least one vertex of e will be v or v', so that e is a bridge of u, so that u doesn't satisfy I2.

Now we have shown that if one of R1 through R4 is false, then there exists a primitive walk that doesn't satisfy at least one of I1 through I3.

$\Leftarrow$: For the other direction, let w be a primitive walk that doesn't satisfy at least one of the conditions I1, I2, and I3. First suppose w is of type P1 or P2. Since P1 primitive walks are C1 circuits and P2 primitive walks are C2 circuits, and since w has either an even chord, a bridge, or an effective crossing, we have shown that one of R1 through R3 must be false for our graph G. So, it remains to consider when w is a P3 primitive walk.

In the next two sections of the proof, "cut vertex" will mean a cut vertex relative to the graph of w.

$\neg$ I1 Now suppose w is a P3 primitive walk of type P3 that doesn't satisfy the condition I1, so that w has an even chord f in one of its cyclic blocks B. First, suppose that B is odd. If we look at $B = (e_1, \ldots, e_n)$ as an odd cycle, then f splits B into one side with an odd number of edges $(e_1, e_2, \ldots, e_k)$ where k is odd. Denote by c' the even cycle $(e_1, \ldots, e_k, f)$, and denote by c_1 the corresponding odd cycle $(e_{k+1}, \ldots, e_n, f)$. Since f is an even chord, c_1 is a cyclic block of some even walk. This means that one of the vertices of c_1 that is not a vertex of f must be the start and the end of some odd path that is vertex-disjoint from c'. Since w is primitive, this vertex v must be a cut vertex of w. Applying Lemma 4.5 to the connected component of $w \setminus \{v\}$ that does not contain f, we get a simple path p, potentially empty, from c_1 to an odd cycle c_2. Then, $c = \left(c_1, p, c_2, p^{-1}\right)$ is a C3 (or C2 if p is empty) circuit. By construction, c and c' are two circuits that share exactly one edge (and no other vertices) contained in a cyclic block of both of them. This contradicts R4.

The second subcase is when B, the cyclic block containing the chord f, is even. If f is an even chord of B when $B = (e_1, \ldots, e_n)$ is considered as a closed even cycle, then we have shown that R1 is not true, with B being the offending circuit. Suppose that f is an odd chord of B when B is considered as a closed even cycle. Then f divides B into two odd cycles $c_1 = (e_1, \ldots, e_k, f)$ and $c'_1 = (e_{k+1}, \ldots, e_n, f)$. As above, one of the vertices v of c_1 that is not a vertex of f must be the start and the end of some odd path. Again, we apply Lemma 4.5 to the part of w that doesn'tcontain

f, using v as the cut vertex to get a simple path p and odd cycle c_2 that make $c = (c_1, p, c_2, p^{-1})$ into a C3 or C2 circuit. Symmetrically, we can do the same for c'_1 to get another C3 or C2 circuit $c' = (c'_1, p', c'_2, p'^{-1})$. By construction, c and c' share exactly one edge f and no other vertices, and f is contained in a cyclic block of both. This shows that R4 is false for our graph.

$\neg$ I2 Suppose w is a P3 primitive walk such that its graph W has a bridge, so that it does not satisfy I2. Then, w has a bridge f connecting two vertices v_i, v_j that lie in different blocks B_1 and B_2, which share at most one vertex by the definition of block, for otherwise $B_1 \cup B_2$ would be a larger biconnected subgraph of W, contradicting the definition of the block. Note that by Theorem 2.4, v_i is either a cut vertex of W, or is a non-cut vertex in a cycle of W — in the latter case, the cycle can be even or odd; the same applies to v_j. We claim there is a C2 or C3 circuit such that f is a bridge. Our strategy is to connect v_i, v_j with a simple path p_0, and then find odd cycles on both ends of p.

We first find a simple path p_0 between v_i, v_j. Let $\mathcal{P}$ be the set of simple paths between v_i, v_j that are contained in W; we claim it is non-empty. But this is true since we can just remove any "loops" that occur in a path connecting the two vertices. We then let p_0 be the simple path in $\mathcal{P}$ with the minimal number of cut vertices of W (not necessarily unique), which exists since the number of cut vertices for any walk in W is a well-defined natural number.

Suppose v_i is a cut vertex of W. Then, applying Lemma 4.5 to the part of the graph W not containing v_j obtained from Lemma 4.4, we get a path p_1 from v_i to some odd cycle c_1. Thus, the path $p = (p_0, p_1)$ goes from v_j, through v_i, and ends at c_1.

Suppose v_i is a non-cut vertex in an even cycle B_1 of W. We claim that B_1 contains a cut vertex of W that is not in p_0. First, B_1 contains at least two cut vertices of W since if not, then B_1 contains no cut vertices, in which case $W = B_1$ and w is not a P3 primitive walk, or B_1 contains one cut vertex, in which case this cut vertex is not a sink of B_1, contradicting that w is primitive by Theorem 2.4. So suppose B_1 contains the cut vertices $x_1, x_2, \ldots, x_n$, appearing in that order such that tracing the path around B_1 goes through x_1, then x_2, etc., until it goes through x_n, and then back to x_1, and such that v_i appears between x_n and x_1. Now if p_0 goes through every cut vertex, then after possible relabeling p_0 goes through x_1, x_2, etc., until it goes through x_n and then out of B_1. But then, the path connecting v_i to x_n and then out of B_1 is a simple path connecting v_i, v_j with fewer cut vertices, contradicting the construction of p_0. Thus, letting x_0 denote the cut vertex in B_1 not contained in p_0, there exists a simple path p_1 connecting x_0 to v_i contained in B_1, for otherwise p_0 would not be simple. Now applying Lemma 4.5 at x_0 as in the case when v_i is a cut vertex gives a path p_2 from x_0 to some odd cycle c_1, and the path $p = (p_0, p_1, p_2)$ goes from v_j, through v_i, and ends at c_1.

Now suppose v_i is a non-cut vertex contained in an odd cycle B_1. We claim we can find a subpath p of p_0 such that it only intersects B_1 once. Orienting p_0 such that it starts at v_i, we can find the last vertex v_1 such that $v_1 \in B_1$. Letting p_1 be the subpath of p starting at v_1, we have the subpath desired. Letting p_2 be the simple path from v_1 to v_i fully contained in B_1, we see that the path $p = (p_1, p_2)$ goes from v_j to an odd cycle B_1 which contains v_i.

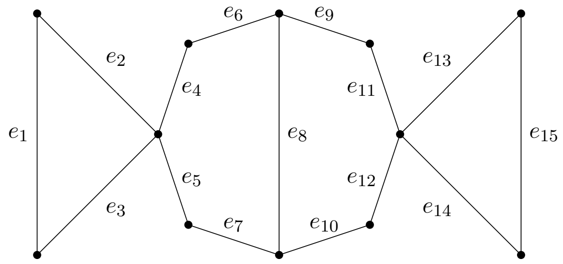
Repeating the same process at v_j, we see that we can thereby construct a C2 or C3 circuit such that f is a bridge between v_i, v_j.

$\neg$ I3 Finally, suppose w is a P3 primitive walk that violates I3, that is, w has an effective crossing in some cyclic block $B = (e_1, \dots, e_n)$ of w. Let $f = (v_1, v_3)$ and $f' = (v_2, v_4)$ be the two odd chords of w that cross effectively in w. These chords divide B up into 4 segments of edges, $s_1 = (e_1, \dots, e_k), s_2 = (e_{k+1}, \dots, e_l), s_3 = (e_{l+1}, \dots, e_m), s_4 = (e_{m+1}, \dots, e_n)$, where the vertex between s_j and s_{j+1} is v_{j+1} and the vertex between s_4 and s_1 is v_1. For each i, if s_i has an even number of edges, then at least one vertex v in s_i that is not one of the v_i must be the start and the end of an odd path that is vertex disjoint from the rest of w. This is the case because f and f' cross effectively, so there must be an odd number of edges along w between where one starts and the other ends.

If B is an odd cyclic block, then the sum of the lengths of the s_i is odd, so either three are odd and one is even or three of them are even and one is odd. In the former case, suppose that s_1 is the even one and let v be a vertex as described above. By Lemma 4.5, there is a path p and an odd cycle c contained in the part of w that does not contain B. But then, (B, p, c, p^{-1}) is a closed even walk that is either a C2 or C3 circuit, and f and f' cross effectively in this walk, contradicting R3. In the latter case, suppose, without loss of generality, that s_1, s_2, s_3 are even and s_4 is odd. Let v and v' be vertices in s_1 and s_2, respectively, that are not one of the v_i, that are the starting and ending points of odd walks in the manner described above. As before, in each case, we can find paths p and p' connecting v and v' to odd cyclic blocks c and c'. Let q be the walk that goes along B connecting v to v' that goes through v_1, v_4, and v_3. Then the walk $(c, p^{-1}, q, p', c', p'^{-1}, q^{-1}, p)$ is a C3 circuit that has f as a bridge, contradicting condition R2.

Finally, consider the case where B is an even cyclic block. The sum of the lengths of the s_i is even, so they are either all odd, all even, or two are even and two are odd. If each of the s_i is odd, then B is a closed even cycle with an effective crossing, negating condition R3. If two are even and two are odd, then without loss of generality, we can say that s_1 is even and s_2 is odd. But then, the chord f is an even chord of B when considered as an even cyclic walk, contradicting R1. If all four are even, then we let v and v' be the vertices in s_1 and s_2, respectively, that are the start and the end of a closed even walk. Proceeding as in the above case, we find a C3 circuit of the form $(c, p^{-1}, q, p', c', p'^{-1}, q^{-1}, p)$ that has f as a bridge, negating condition R2. ∎

Remark 4.8. The previous theorems suggest that questions of robustness for a graph G, which is naturally a question about the Universal Gröbner Basis, can be reduced to a question about the Graver Basis, and in turn, to one about the circuits of G. In light of this, it is a natural question to ask if it is true that G is robust (that is, if the primitive walks of G are precisely the indispensable walks of G) if and only if the circuits of G are precisely the indispensable walks of G. It turns out that both directions are false, as demonstrated by the following graphs.



In this graph, the Graver basis consists of the following binomials:

$B_1 = e_4e_7e_8e_{12}^2e_{15} - e_5e_6e_{10}^2e_{13}e_{14}$, $B_7 = e_1e_4^2e_8e_9e_{12} - e_2e_3e_6^2e_{10}e_{11}$,

$B_2 = e_4e_7e_9e_{12} - e_5e_6e_{10}e_{11}$, $B_8 = e_1e_4^2e_9^2e_{13}e_{14} - e_2e_3e_6^2e_{11}^2e_{15}$,

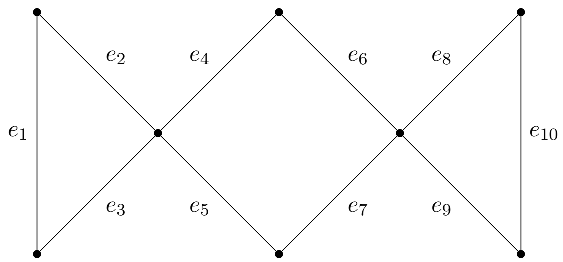
$B_3 = e_1e_5^2e_{10}^2e_{13}e_{14} - e_2e_3e_7^2e_{12}^2e_{15}$, $B_9 = e_1e_5^2e_8e_{10}e_{11} - e_2e_3e_7^2e_9e_{12}$,

$B_4 = e_8e_{11}e_{12}e_{15} - e_9e_{10}e_{13}e_{14}$, $B_{10} = e_4e_7e_9^2e_{13}e_{14} - e_5e_6e_8e_{11}^2e_{15}$,

$B_5 = e_1e_4e_5e_8 - e_2e_3e_6e_7$, $B_{11} = e_1e_5^2e_8^2e_{11}^2e_{15} - e_2e_3e_7^2e_9^2e_{13}e_{14}$,

$B_6 = e_1e_4^2e_8^2e_{12}^2e_{15} - e_2e_3e_6^2e_{10}^2e_{13}e_{14}$, $B_{12} = e_1e_4e_5e_9e_{10}e_{13}e_{14} - e_2e_3e_6e_7e_{11}e_{12}e_{15}$.

The circuits are precisely the first 11 binomials, as are the indispensable binomials. The binomial B_{12} is primitive and an element of the Universal Gröbner Basis, but not indispensable. This is a counterexample to the backwards direction of the proposed statement above, since the set of circuits is the same as the set of indispensable walks, but the graph is not robust. This also gives an example of a graph that has a unique minimal generating set, but is not robust. To demonstrate a counterexample to the forward direction of the proposed statment, we offer the following robust graph.



In this graph, the Graver basis consists of the following binomials:

$B_1 = e_4e_7 - e_5e_6$, $B_3 = e_1e_5^2e_8e_9 - e_2e_3e_7^2e_{10}$,

$B_2 = e_1e_4^2e_8e_9 - e_2e_3e_6^2e_{10}$, $B_4 = e_1e_4e_5e_8e_9 - e_2e_3e_6e_7e_{10}$.

All four of these are elements of the Universal Gröbner Basis, and they are precisely the indispensable binomials of this graph. However, B_4 is not a circuit. In this case, the graph is robust, but there is a noncircuit indispensable binomial.

5. Applications to Low-Degree Graph Ideals

The characterization given in Theorem 4.7 takes on a particularly simple form in the case where the ideal is generated by quadratic binomials.

Proposition 5.1. *If G is a simple graph and I_G is robust and minimally generated by quadratics, then any two circuits of G either share no edges or exactly two edges of opposite parity.*

Proof. It suffices to restrict the characterization in Theorem 4.7 to this special case. By Theorem 3.2, we know that all primitive elements are minimal generators; in particular, all the circuits of G are even cycles of length 4, which we will call "squares".

Now suppose that there exist two squares of G that share one or three edges. If they share three edges, then G would not be simple, contradicting our assumptions on G. Next, suppose they share one edge. Then, our squares are as follows

$$c_1\colon v_1 \longrightarrow v_2 \xrightarrow{f} v_3 \longrightarrow v_4 \longrightarrow v_1$$

$$c_2\colon w_1 \longrightarrow v_2 \xrightarrow{f} v_3 \longrightarrow w_4 \longrightarrow w_1$$

where f is the shared edge, and the larger circuit

$$c_3\colon v_1 \longrightarrow v_2 \longrightarrow w_1 \longrightarrow w_4 \longrightarrow v_3 \longrightarrow v_4 \longrightarrow v_1$$

has f as an even chord, contradicting robustness by Theorem 4.7.

Now consider when two squares share two edges. If they share two edges of the same parity, then all 4 of their vertices are the same, and they must be one of the following forms:

$$c_1\colon v_1 \xrightarrow{f} v_2 \xrightarrow{g} v_3 \xrightarrow{f'} v_4 \xrightarrow{h} v_1$$

$$c_2\colon v_1 \xrightarrow{f} v_2 \xrightarrow{g'} v_3 \xrightarrow{f'} v_4 \xrightarrow{h'} v_1$$

or

$$c_1\colon v_1 \xrightarrow{f} v_2 \xrightarrow{g} v_3 \xrightarrow{f'} v_4 \xrightarrow{h} v_1$$

$$c_2\colon v_1 \xrightarrow{f} v_2 \xrightarrow{g'} v_4 \xrightarrow{f'} v_3 \xrightarrow{h'} v_1$$

where f and f' are the shared edges. In the first case, our graph is not simple because there are two distinct edges g, g' (h, h', respectively) connecting v_2 and v_3 (v_1 and v_4, respectively), and in the second case, the circuit c_1 has an effective crossing g'. ∎

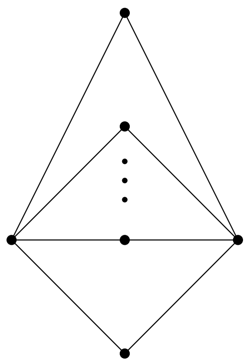


Figure 8: A typical robust quadratic graph ideal.

For a general robust toric ideal, we say it is irreducible if the minimal generators cannot be partitioned into two sets of polynomials in disjoint sets of variables. A result from [2] shows that irreducible robust quadratic ideals that are not principal are generated by the 2×2 minors of a generic $2 \times n$ matrix. Since we can associate graphs that are of the type in Figure 8 with $2 \times n$ matrices, we have the following corollary.

Corollary 5.2. *All non-principal irreducible robust toric ideals generated by quadratics are graph ideals that arise from the family of graphs given in Figure 8.*

Finally, to show the wide variety of possibilities for robust graphs, we have computed the set of connected robust graphs on seven vertices in Figure 9. To avoid trivialities, we assume the graph ideals have full support in their edge ring. The graphs are partitioned and labeled so that graphs that give isomorphic ideals are in the same partition. Notice that all the graphs yield irreducible robust toric ideals except for the second one.

6. Concluding Remarks

In light of Theorem 3.2, it is natural to ask whether we can generalize this statement to toric ideals not arising from graphs. Our proof of Theorem 3.2 relied heavily on graph-theoretic arguments, but perhaps there is a more algebraic proof. Whether or not this theorem generalizes to all toric ideals remains an open question.

Question 6.1. If I_A is a robust toric ideal, is I_A minimally generated by its Graver basis?

7. Appendix

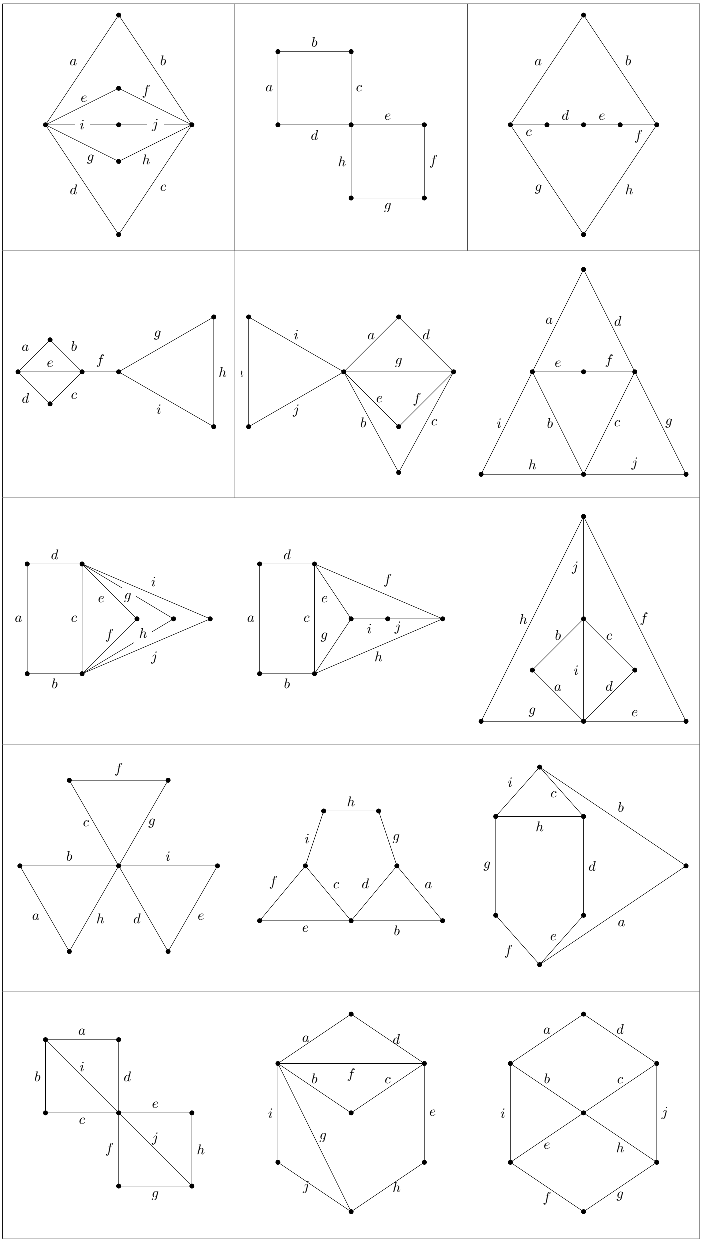


Figure 9: All connected robust graphs G on 7 vertices such that the ideal I_G has full support in its edge ring, divided up into isomorphism classes of I_G.

Acknowledgments. This project was completed at the UC Berkeley Geometry, Topology, and Operator Algebras RTG Summer Research Program for Undergraduates in 2013 and was partially funded by the NSF award number 0838703. We are grateful to UC Berkeley for providing wonderful facilities to conduct this research. We also thank the anonymous referees for their very detailed and helpful suggestions.

References

1. Bogart, T., Hemmecke, R., Petrović, S.: Universal Gröbner bases of colored partition identities. Exp. Math. 21(4), 395–401 (2012)
2. Boocher, A., Robeva, E.: Robust toric ideals. J. Symbolic Comput. 68(1), 254–264 (2015)
3. Gross, E., Petrović, S.: Combinatorial degree bound for toric ideals of hypergraphs. Internat. J. Algebra Comput. 23(6), 1503–1520 (2013)
4. Ohsugi, H., Hibi, T.: Toric ideals generated by quadratic binomials. J. Algebra 218(2), 509–527 (1999)
5. Reyes, E., Tatakis, C., Thoma, A.: Minimal generators of toric ideals of graphs. Adv. Appl. Math. 48(1), 64–78 (2012)
6. Sturmfels, B.: Gröbner Bases and Convex Polytopes. University Lecture Series 8. American Mathematical Society, Providence, RI (1996)
7. Tatakis, C., Thoma, A.: On the universal Gröbner bases of toric ideals of graphs. J. Combin. Theory Ser. A 118(5), 1540–1548 (2011)
8. Villarreal, R.H.: Rees algebras of edge ideals. Comm. Algebra 23(9), 3513–3524 (1995)