Geometric Invariants on Monomial Curves

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What is a Monomial Curve?

We begin with the following example:

Example (A Rational Normal Curve)

Consider the following parametrization of a surface:

$$\mathbf{R}^2 \hookrightarrow \mathbf{R}^5$$
$$(s,t) \mapsto (s^4, s^3t, s^2t^2, st^3, t^4).$$

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Main research goal: Consider what happens to certain geometric invariants when we forget about one or more of the coordinates in the image.

What is a Monomial Curve? (cont.)

Let's now make this more precise.

Definition (Monomial Curve)

A *monomial curve* of degree d with parameters $\mathcal{A} = \{0, a_1, \ldots, a_c, d\}$ such that $0 < a_1 < \cdots < a_c < d$ is the curve defined by the parametrization

$$\mathbf{P}^1 \hookrightarrow \mathbf{P}^{c+1}$$
$$(s,t) \mapsto (s^d, s^{d-a_1}t^{a_1}, s^{d-a_2}t^{a_2}, \ldots, s^{d-a_c}t^{a_c}, t^d).$$

In the previous example, the parameters of the degree-4 monomial curve were $\mathcal{A} = \{0, 1, 2, 3, 4\}$.

A Translation into Algebra

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- Given the parameters $\mathcal{A} = \{0, a_1, \ldots, a_c, d\}$ where $a_0 := 0, a_{c+1} := d$, we can consider the following map of polynomial rings:

$$\phi\colon \mathbf{R}[x_0, x_1, \ldots, x_{c+1}] \to \mathbf{R}[s, t]$$
$$x_i \mapsto s^{d-a_i} t^{a_i}.$$

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- In particular, we want to look at the elements of the kernel $I_{\mathcal{A}}$ of this map. We can study the geometric properties of the curve by studying the algebraic properties of the kernel.

An Example

Let's return to the first example. Our parameters are $\mathcal{A} = \{0, 1, 2, 3, 4\}$, so we define the map

$$\begin{aligned}
\phi \colon \mathbf{R}[x_0, x_1, x_2, x_3, x_4] &\to \mathbf{R}[s, t] \\
x_0 &\mapsto s^4 \\
x_1 &\mapsto s^3 t \\
x_2 &\mapsto s^2 t^2 \\
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$$x_4 \mapsto t^4.$$

Then, $ker\phi$ is generated by the following binomials:

$$x_3^2 - x_2x_4$$
$$x_2x_3 - x_1x_4$$
$$x_1x_3 - x_0x_4$$
$$x_2^2 - x_0x_4$$
$$x_1x_2 - x_0x_3$$
$$x_1^2 - x_0x_2.$$

Notice how they all have degree 2.

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The largest of these generators has degree 3. In general, the more points we remove, the larger the degree of the generators. We want to try to bound the size of these generators in some way.

Regularity

- There is a well-defined algebro-geometric invariant that bounds the degrees of the generators of our kernel above, which is called *regularity*.

Example

The regularity of the curve with parameters $\mathcal{A} = \{0, 1, 2, 3, 4\}$ is 2.
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- Regularity is generally hard to compute because of its technical definition.
- In the case of monomial curves, we can compute this combinatorially.

Computing Regularity

Let $\mathcal{A} = \{0, a_1, \ldots, a_c, d\}$ be our set of parameters.

An Algorithm

1. Compute $\mathfrak{M}_i$, the natural numbers that can be *minimally* expressed as a sum of i elements of $\mathcal{A} \setminus \{0\}$.
2. Eventually $\#\mathfrak{M}_i = d$ and $\mathfrak{M}_i \subseteq d + \mathfrak{M}_{i-1}$.
3. The first i when this occurs is the regularity.

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Example ($\mathcal{A} = \{0, 1, 3, 4\}$)

$\mathfrak{M}_0 = \{0\}$
$\mathfrak{M}_1 = \{1, 3, 4\}$
$\mathfrak{M}_2 = \{2, 5, 6, 7, 8\}$
$\mathfrak{M}_3 = \{9, 10, 11, 12\}$
Regularity $= 3$

A Harder Example

Example ($\mathcal{A} = \{0, 2, 5, 7\}$)

1. Compute the $\mathfrak{M}_i$:

$$\begin{aligned}
\mathfrak{M}_0 &= \{0\} \\
\mathfrak{M}_1 &= \{2, 5, 7\} \\
\mathfrak{M}_2 &= \{4, 9, 10, 12, 14\} \\
\mathfrak{M}_3 &= \{6, 11, 15, 16, 17, 19, 21\} \\
\mathfrak{M}_4 &= \{8, 13, 18, 20, 22, 23, 24, 26, 28\} \\
\mathfrak{M}_5 &= \{25, 27, 29, 30, 31, 33, 35\} \\
&\vdots
\end{aligned}$$

2. $\#\mathfrak{M}_5 = 7$ and $\mathfrak{M}_5 \subseteq 7 + \mathfrak{M}_4$
3. Regularity $= 5$

The General Case

Question

Even if we can compute regularity in specific cases, is there a bound on regularity for *all* monomial curves, in terms of its parameters?

Bounds on Regularity

Recall $\mathcal{A} = \{0, a_1, \ldots, a_c, d\}$, and $I_{\mathcal{A}}$ is the kernel of the map from before.

Bounding regularity is a tough problem. Using fancy cohomological machinery, Gruson, Lazarsfeld, and Peskine found the following bound:

GLP Bound

$$\operatorname{reg} I_{\mathcal{A}} \leq d - c + 1$$

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With similar techniques, L'vovsky found the following improvement:

L'vovsky Bound

$$\operatorname{reg} I_{\mathcal{A}} \leq \max_{1 \leq i < j \leq c+1} \{a_i - a_{i-1} + a_j - a_{j-1}\}$$

Results

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- We were able to show the following improvement:

Our result

$$\operatorname{reg} I_{\mathcal{A}} \leq d + 2 - \#\{i, j \in \mathfrak{M}_1 \cup \mathfrak{M}_2 \mid i - j = d\}.$$

- This is an improvement since $\#\{i, j \in \mathfrak{M}_1 \cup \mathfrak{M}_2 \mid i - j = d\} \geq c + 1$.

A Comparison

Let $\mathcal{A} = \{0, 1, 5, 8, 14, 19\}$.

GLP Bound

$19 - 4 + 1 = 16$

L'vovsky Bound

$(14 - 8) + (19 - 14) = 11$

Our Bound

$19 + 2 - 6 = 15$

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Our Bound

$19 + 2 - 6 = 15$

But, the regularity is 5. The maximum degree of a generator of $I_{\mathcal{A}}$ is 4.

Further Directions

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- Thanks!