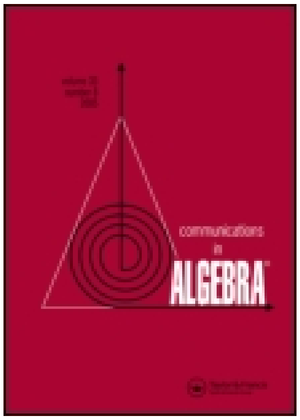

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DIMENSIONS OF FORMAL FIBERS OF HEIGHT ONE PRIME IDEALS

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Let T be a complete local (Noetherian) ring with maximal ideal M, P a nonmaximal ideal of T, and $C = \{Q_1, Q_2, \ldots\}$ a (nonempty) finite or countable set of nonmaximal prime ideals of T. Let $\{p_1, p_2, \ldots\}$ be a set of nonzero regular elements of T, whose cardinality is the same as that of C. Suppose that $p_i \in Q_j$ if and only if $i = j$. We give conditions that ensure there is an excellent local unique factorization domain A such that A is a subring of T, the maximal ideal of A is $M \cap A$, the $(M \cap A)$-adic completion of A is T, and so that the following three conditions hold: (1) $p_i \in A$ for every i; (2) $A \cap P = (0)$, and if J is a prime ideal of T with $J \cap A = (0)$, then $J \subseteq P$ or $J \subseteq Q_i$ for some i; (3) for each i, p_iA is a prime ideal of A, $Q_i \cap A = p_iA$, and if J is a prime ideal of T with $J \not\subseteq Q_i$, then $J \cap A \neq p_iA$.

Key Words: Completions; Excellent rings; Formal fibers; Local rings.

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1. INTRODUCTION

In this article, we explore the relationship between a local ring A and its completion by studying how certain prime ideals of the completion of A intersect with the ring A. Let A be a local (Noetherian) ring with maximal ideal M, and let $\widehat{A}$ denote the M-adic completion of A. We are interested in the map $\Psi: \operatorname{Spec}\widehat{A} \longrightarrow \operatorname{Spec} A$ given by $Q \longrightarrow Q \cap A$. Since the extension $A \longrightarrow \widehat{A}$ is faithfully flat, we know that Ψ is surjective. In particular, we assume that A is an integral domain and focus on the inverse image under Ψ of (0) and a countable number of height one prime ideals of A. While the inverse image of (0) has been studied in, for example, [1, 7, 9, 10], and the inverse image of height one prime ideals has been studied separately in, for example, [2, 3], the only result we know of in which the relationship between the inverse image of (0) and the inverse image of infinitely many height one prime ideals has been studied is in [8]. After some preliminary

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definitions and results, we describe the main theorem in [8] and explain how it relates to the main result in this article.

Let A be a local ring with maximal ideal M and P a prime ideal of A. Following Matsumura in [9], we define the formal fiber ring of A at P to be $\widehat{A} \otimes_A k(P)$, where $k(P)$ is the field A_P/PA_P. The formal fiber of A at P is defined to be $\mathrm{Spec}(\widehat{A} \otimes_A k(P))$. Since there is a one-to-one correspondence between elements in the formal fiber of A at P and prime ideals Q of $\widehat{A}$ satisfying $Q \cap A = P$, we will abuse notation and say that such a prime ideal Q is in the formal fiber of A at P. If A is an integral domain with quotient field K, we define $\alpha(A)$ to be the Krull dimension of the ring $\widehat{A} \otimes_A K$. In other words, $\alpha(A)$ is the dimension of the formal fiber ring at (0). Heinzer, Rotthaus, and Sally have informally asked the question.

Question 1.1. If A is an excellent local integral domain with $\alpha(A) > 0$, then is the set of height one prime ideals p of A satisfying $\alpha(A/p) = \alpha(A)$ a finite set?

In Example 2.26, we provide an example showing that, in fact, this set can be infinite. In other words, our result shows that we can control the relationship between the dimension of the fiber ring at (0) and the dimension of fiber rings over infinitely many height one prime ideals of an excellent local integral domain A.

The main result in [8] answers Question 1.1 when we do not require that A be excellent. In particular, let T be a complete local unique factorization domain, p a nonmaximal prime ideal of T, and F a set of nonmaximal prime ideals of T. Conditions are given in Theorem 23 of [8] to ensure that there exists a local unique factorization domain A such that $\widehat{A} = T$, $p \cap A = (0)$, $Q \cap A \neq (0)$ for all prime ideals Q of T such that $\operatorname{ht} Q > \operatorname{ht} p$, and $A \cap q = z_q A$ for all $q \in F$, where z_q is a nonzero prime element of T. As an example, the author lets $T = \mathbb{C}[[X_1, X_2, \ldots, X_n]]$, where $n > 3$ and $p = (X_1, \ldots, X_{n-2})$, F an infinite set of height $n-1$ prime ideals of T such that $|F| < |\mathbb{C}|$. Now, T, p, and F satisfy the conditions of Theorem 23 in [8], so there is a local unique factorization domain A with $\widehat{A} = T$ and such that $\alpha(A) = n - 2 = \alpha(A/(q \cap A))$ for every $q \in F$. Since $A \cap q = z_q A$ for every $q \in F$, we have that $q \cap A$ is a prime ideal for every $q \in F$, and so this example answers Question 1.1 if we drop the condition that A need be excellent. In this article, we provide a similar example (see Example 2.26), but the A that we give is more difficult to construct because it is, in fact, excellent. Although the basic outline for the construction in our article is similar to the construction in [8], the technical details are quite different.

Theorem 2.24 is our main result and not only provides an answer for Question 1.1, but it also gives an excellent unique factorization domain whose formal fibers have other nonstandard properties. Again, let A be a local ring with completion $\widehat{A}$. Suppose Q is a prime ideal of $\widehat{A}$ satisfying the property that if J is a prime ideal of $\widehat{A}$ with $Q \cap A = J \cap A = P$, then $J \subseteq Q$. Then we say that the formal fiber of A at P is local with maximal ideal Q. For standard excellent local domains, local formal fibers seem to be very rare. For example, for the ring $R = \mathbb{C}[x_1, x_2, \ldots, x_n]_{(x_1, x_2, \ldots, x_n)}$, where $n > 2$, no prime ideals of height less than $n-1$ have a local formal fiber. The excellent local unique factorization domain A we construct in Theorem 2.24, however, satisfies the very unusual property that *all* of its prime ideals have a local formal fiber except for the zero ideal. Moreover, we are able to describe in detail all of the formal fibers of A, also extremely unusual. Indeed, there are still many open questions about the formal fibers of even the standard excellent local ring $R = \mathbb{C}[x_1, x_2, \ldots, x_n]_{(x_1, x_2, \ldots, x_n)}$. In fact, it was only recently, in [4], that Heinzer, Rotthaus, and Wiegand proved that for the ring $R = \mathbb{C}[x_1, x_2, \ldots, x_n]_{(x_1, x_2, \ldots, x_n)}$ every maximal element of the formal fiber of R at (0) has height $n-1$.

We now describe our main result, Theorem 2.24, in detail. Suppose that T is a complete local ring with maximal ideal M. Let P be a nonmaximal prime ideal of T and $C = \{Q_1, Q_2, \ldots\}$ a (nonempty) countable or finite set of nonmaximal prime ideals of T. Let $\{p_1, p_2, \ldots\}$ be a set of nonzero regular elements of T whose cardinality is the same as the cardinality of C. Suppose also that $p_i \in Q_j$ if and only if $i = j$. Let R_0 be the prime subring of T and $R_i = R_0[p_1, p_2, \ldots, p_i]$ for $i = 1, 2, \ldots$. Define $S = \bigcup_{i=0}^{\infty} R_i$ if C is infinite and $S = \bigcup_{i=0}^{k} R_i$ if C contains $k < \infty$ elements. Suppose $S \cap P = (0)$, $S \cap P' = (0)$ whenever P' is an associated prime ideal of T and for each i, $(Q_i \backslash p_i T)S \cap S = \{0\}$, where $(Q_i \backslash p_i T)S = \{qs \mid q \in Q_i, q \notin p_i T \text{ and } s \in S\}$. Assume that the following conditions hold:

(1) For each i if $Q \in \operatorname{Ass}(T/p_i T)$, we have $Q \subseteq Q_i$;
(2) T is a UFD;
(3) $|T| = |T/M|$;
(4) T contains the rationals;
(5) T_P is a regular local ring and for all i, T_{Q_i} and $(T/p_i T)_{Q_i}$ are regular local rings.

We show that there exists an excellent local unique factorization domain $A \subseteq T$ such that:

(1) $p_i \in A$ for all i;
(2) $\widehat{A} = T$;
(3) $A \cap P = (0)$ and if J is a prime ideal of T with $J \cap A = (0)$, then $J \subseteq P$ or $J \subseteq Q_i$ for some i;
(4) For each i, $p_i A$ is a prime ideal in A and has a local formal fiber with maximal ideal Q_i.

We go on to describe the formal fibers of A in detail. We show that the formal fiber of A at $p_i A$ is the set $\{J \in \operatorname{Spec} T \mid J \subseteq Q_i, \text{ and } p_i \in J\}$, the formal fiber of A at (0) is the set $\{P\} \cup \{J \in \operatorname{Spec} T \mid J \subseteq Q_i, \text{ for some } i \text{ and } p_i \notin J\}$, and the formal fiber of A at Q, where Q is a nonzero prime ideal of A and $Q \neq p_i A$ for all i is the set $\{QT\}$. It follows that all formal fibers of A are local except the one at (0). We then use Theorem 2.24 to produce Example 2.26, which answers Question 1.1.

In Theorem 2.20 we prove an analogous result to Theorem 2.24, where the ring A we construct is not excellent. Also, in Theorems 2.21 and 2.25, we show that if the set C is finite, we can remove the requirement that T be a unique factorization domain.

The idea of the construction of our excellent unique factorization domain A is to first assume that a special subring of T, called a *PC-subring* (see Definition 2.1) exists. Call this subring R. In particular, we guarantee that R contains p_i for every i. We then adjoin elements of T to R to build a chain $\{R_\alpha\}$ of *PC*-subrings satisfying the following properties:

(1) $R_\alpha \cap P = (0)$ for every α;
(2) For all i and for all α, $Q_i \cap R_\alpha = p_i R_\alpha$;

(3) For all α, we have that if I is a finitely generated ideal of R_α, then $IT \cap R_\alpha = I$;
(4) If J is a prime ideal of T such that $J \not\subseteq P$ and $J \not\subseteq Q_i$ for every i, then given an element, $u + J$, of T/J, there exists an R_α that contains a nonzero element of $u + J$.

Our excellent unique factorization domain A will be the union of the R_α's. Condition (1) will guarantee that $A \cap P = (0)$, so that we have P in the formal fiber of A at (0). Condition (2) will ensure that $A \cap Q_i = p_i A$, so that we have Q_i in the formal fiber of A at $p_i A$. From conditions (3) and (4), we get that the map $A \longrightarrow T/M^2$ is onto and that $IT \cap A = I$ for every finitely generated ideal I of A. This is enough (see Proposition 2.8) to conclude that the completion of A is T. Condition (4) also gives us that if J is a prime ideal of T with $J \not\subset P$ and $J \not\subseteq Q$ for all $Q \in C$, then the map $A \longrightarrow T/J$ is surjective. We will use this to show that all formal fibers of A are as desired and that A is excellent. After proving Theorem 2.24, we show that there are many complete local rings T for which PC-subrings, in fact, do exist.

2. THE CONSTRUCTION

Before we begin constructing our ring A, we comment on the notation used in this article. When we say that a ring is local, Noetherian is implied. A quasi-local ring is one that has exactly one maximal ideal but that may not be Noetherian. To denote a local ring T with maximal ideal M, we use the notation (T, M). We will use the standard abbreviation UFD to denote a unique factorization domain. When we refer to our final ring A, we mean the ring A from Theorem 2.24.

Definition 2.1. Let (T, M) be a complete local ring, P be a nonmaximal prime ideal of T, and $C = \{Q_1, Q_2, \dots\}$ a (nonempty) countable or finite set of nonmaximal prime ideals of T. Let $\{p_1, p_2 \dots\}$ be a set of nonzero regular elements of T whose cardinality is the same as the cardinality of C. Suppose also that $p_i \in Q_j$ if and only if $i = j$. Let $(R, R \cap M)$ be an infinite quasi-local subring of T such that $p_i \in R$ for every $i = 1, 2, \dots$ and such that the following conditions hold:

(1) $|R| < |T|$;
(2) $R \cap P = (0)$ and if P' is an associated prime ideal of T, then $R \cap P' = (0)$;
(3) For each i, $(Q_i \backslash p_i T)R \cap R = \{0\}$, where $(Q_i \backslash p_i T)R = \{qr \mid q \in Q_i \backslash p_i T, r \in R\}$.

Then we call the ring R a PC-subring of T with respect to the set $\{p_1, p_2, \dots\}$. If the set $\{p_1, p_2, \dots\}$ is clear from the context, we will simply say that R is a PC-subring of T.

Suppose that (T, M), C, P, and $\{p_i\}$ are as in Definition 2.1. Then the Krull dimension of T is at least one and so by Lemma 2.2 in [3], we have that $|T| \geq |\mathbb{R}|$, where $\mathbb{R}$ denotes the real numbers. We also have that by condition (2), R contains no zero-divisors of T.

The idea for property (3) of PC-subrings is inspired by the definition of pT-complement avoiding subrings of T in [2]. Indeed, to show that a certain subring of T satisfies condition (3) of PC-subrings, we will often use ideas from proofs in [2].

The following lemma shows that *PC*-subring properties are preserved under localization.

Lemma 2.2. *Let* (T, M), C, P, *and* $\{p_i\}$ *be as in Definition* 2.1. *Let* R *be a subring of* T *satisfying all conditions of PC-subring except that it is not necessarily a quasi-local ring. Then* $R_{(M\cap R)}$ *is a PC-subring of* T.

Proof. Conditions (1) and (2) of *PC*-subrings clearly hold for $R_{(M\cap R)}$. So now suppose that for some i, $s \in (Q_i \backslash p_i T)R_{(M\cap R)} \cap R_{(M\cap R)}$. We can then write $s = f/g = qf'/g'$ with $f, g, f', g' \in R$, $g, g' \notin M$, and $q \in Q_i \backslash p_i T$. Since R satisfies condition (2) of *PC*-subrings, it contains no zero-divisors of T. It follows that $g'f = qgf' \in (Q_i \backslash p_i T)R \cap R = \{0\}$. Since $g' \neq 0$, we have $f = 0$, and so $s = 0$, as desired. □

In the next lemma we show that taking certain unions of *PC*-subrings will preserve the *PC*-subring properties.

Lemma 2.3. *Let* (T, M), C, P, *and* $\{p_i\}$ *be as in Definition* 2.1. *Let* Ω *be a well-ordered set, and let* $\{R_\alpha \mid \alpha \in \Omega\}$ *be a set of PC-subrings indexed by* Ω *with the property that* $R_\alpha \subseteq R_\beta$ *for all* α, β *satisfying* $\alpha < \beta$. *Let* $S = \bigcup_{\alpha\in\Omega} R_\alpha$. *Then* S *satisfies all properties of PC-subrings except for possibly condition* (1). *Moreover, if* $|R_\alpha| \leq \lambda$ *for all* $\alpha \in \Omega$, *then* $|S| \leq \lambda \cdot \sup\{|\Omega|, \aleph_0\}$. *In particular, if* $|\Omega| \leq \lambda$, $|R_\alpha| \leq \lambda$ *for all* α *and* $|R_\alpha| = \lambda$ *for some* α *we have* $|S| = \lambda$.

Proof. The cardinality conditions are clear. Condition (2) of *PC*-subrings holds for S since it holds for every R_α. We now show that property (3) in the definition of *PC*-subring is satisfied. Suppose that for some i, we have $f \in (Q_i \backslash p_i T)S \cap S$. Then $f = qg$ with $q \in (Q \backslash p_i T)$ and $g \in S$. So $g, f \in R_\alpha$ for some α, and hence $f = qg \in (Q \backslash p_i T)R_\alpha \cap R_\alpha = (0)$, which shows $f = 0$ as required. □

Recall that we want our final ring A to satisfy the property that $A \cap Q_i = p_i A$ for all i. While we cannot maintain this property at every step, we can show that when the property is not satisfied for a *PC*-subring R, we can build a larger *PC*-subring S that does satisfy $S \cap Q_i = p_i S$ for all i. The next three lemmas show how to do this. This idea was inspired by Lemmas 3.2, 3.4 and 3.5 from [3].

Lemma 2.4. *Let* (T, M), C, P, *and* $\{p_i\}$ *be as in Definition* 2.1. *Suppose* $(R, R\cap M)$ *is a PC-subring of* T *and let* $\{c_1, c_2, \dots\}$ *be a set of elements of* T *whose cardinality is the same as that of* C. *Suppose also that* $c_i \in p_i T \cap R$ *for every* i. *Then there exists a PC-subring* S *of* T *such that* $R \subseteq S \subseteq T$, $c_i \in p_i S$ *for each* i *and* $|R| = |S|$.

Proof. Since $c_1 \in p_1 T \cap R$, we have that $c_1 = p_1 u$ for some $u \in T$. We claim that $S_1 = R[u]_{(R[u]\cap M)}$ is a *PC*-subring. Note that we have $c_1 \in p_1 S_1$, and $|S_1| = |R|$.

Condition (1) for *PC*-subrings is clearly satisfied by $R[u]$. Now suppose Q is a prime ideal of T satisfying $Q \cap R = (0)$. We claim that $Q \cap R[u] = (0)$. Suppose $f \in Q \cap R[u]$. Then $f = r_n u^n + \cdots + r_1 u + r_0$ where $r_i \in R$. Now, $p_1^n f = r_n c_1^n + \cdots + r_1 c_1 p_1^{n-1} + r_0 p_1^n \in Q \cap R = (0)$. Since p_1 is a regular element of T, we have that $f = 0$.

Now suppose that for some i, we have $f \in (Q_i \backslash p_i T)R[u] \cap R[u]$. Then $f = qs = z$, where $q \in Q_i \backslash p_i T$ and $s, z \in R[u]$. As in the above paragraph, we can show that $p_1^n s$ and $p_1^m z$ are elements of R for appropriate integers n and m. Now let $N = m + n$, and note that $p_1^N f = p_1^N qs = p_1^N z \in (Q_i \backslash p_i T)R \cap R = \{0\}$. As p_1 is a regular element of T, we have that $f = 0$. Now using Lemma 2.2, we have that S_1 is a PC-subring of T.

Note that $c_i \in p_i T \cap R \subseteq p_i T \cap S_1$ for every i and $c_1 \in p_1 S_1$. Now construct a PC-subring S_2 of T using the above argument replacing R by S_1, p_1 by p_2, and c_1 by c_2 so that $S_1 \subseteq S_2$, $|S_2| = |R|$, $c_i \in p_i T \cap S_2$ for every i and $c_2 \in p_2 S_2$. Continue to construct S_j for $j = 1, 2 \ldots$ so that $c_j \in p_j S_j$. Define $S = \bigcup_{i=1}^{\infty} S_i$ if C is infinite, and define $S = S_k$ if C contains $k < \infty$ elements. Then using Lemma 2.3 in the infinite case, we have that S satisfies $|S| = |R|$ and is a PC-subring. It is clear that $R \subseteq S \subseteq T$. To see that $c_i \in p_i S$, just note that $c_i \in p_i S_i \subseteq p_i S$. □

Definition 2.5. Let Ω be a well-ordered set and $\alpha \in \Omega$. We define $\gamma(\alpha) = \sup\{\beta \in \Omega \mid \beta < \alpha\}$.

Lemma 2.6. *Let (T, M), C, P, and $\{p_i\}$ be as in Definition 2.1. Suppose $(R, R \cap M)$ is a PC-subring of T. Then there exists a PC-subring S of T with $|S| = |R|$ such that $R \subseteq S \subseteq T$ and $p_i T \cap R \subseteq p_i S$ for each i.*

Proof. Let $\Omega = p_1 T \cap R$ and note that $|\Omega| \leq |R|$. Well order Ω letting 0 denote the first element and define $R_0 = R$. Note that as $p_1 R \subseteq p_1 T \cap R$ and R is infinite, we have that Ω has no maximal element. We will inductively define R_α for every $\alpha \in \Omega$. Let $\alpha \in \Omega$, and assume that for all $\beta < \alpha$, R_β has been defined and satisfies $|R_\beta| = |R|$ and $\delta \in p_1 R_\beta$ for all $\delta < \beta$. We now work to define R_α.

If $\gamma(\alpha) < \alpha$, then obtain R_α from $R_{\gamma(\alpha)}$ using Lemma 2.4 with $c_1 = \gamma(\alpha)$. Then R_α is a PC-subring of T and $|R_\alpha| = |R_{\gamma(\alpha)}| = |R|$. Since $\gamma(\alpha) \in p_1 R_\alpha$ and $R_{\gamma(\alpha)} \subseteq R_\alpha$, using the induction hypothesis, we see that $\delta \in p_1 R_\alpha$ for all $\delta < \alpha$.

On the other hand, if $\gamma(\alpha) = \alpha$ define $R_\alpha = \bigcup_{\beta < \alpha} R_\beta$. Then by Lemma 2.3, R_α is a PC-subring and $|R_\alpha| = |R|$. We also have, by induction, that $\delta \in p_1 R_\alpha$ for all $\delta < \alpha$.

Now, let $S_1 = \bigcup_{\alpha \in \Omega} R_\alpha$. Then by Lemma 2.3, S_1 is a PC-subring with $|S_1| = |R|$. If $r \in p_1 T \cap R$, then $r = \gamma(\alpha)$ for some α in Ω with $\gamma(\alpha) < \alpha$. By construction $r \in p_1 R_\alpha \subseteq p_1 S_1$, and so $p_1 T \cap R \subseteq p_1 S_1$.

Now, repeat the above construction replacing R by S_1 and p_1 by p_2 to construct a PC-subring S_2 so that $S_1 \subseteq S_2$, $|S_2| = |R|$, and $p_2 T \cap S_1 \subseteq p_2 S_2$. In this manner, define S_k for $k = 2, 3 \ldots$ so that $S_{k-1} \subseteq S_k$, $|S_k| = |R|$, and $p_k T \cap S_{k-1} \subseteq p_k S_k$. Now let $S = \bigcup_{k=1}^{\infty} S_k$ if C is infinite, and let $S = S_n$ if C contains $n < \infty$ elements. By Lemma 2.3, S is a PC-subring and $|S| = |R|$. Note that if $f \in p_i T \cap R$, then $f \in p_i T \cap S_{i-1} \subset p_i S$. □

Lemma 2.7. *Let (T, M), C, P, and $\{p_i\}$ be as in Definition 2.1. Suppose $(R, R \cap M)$ is a PC-subring of T. Then there exists a PC-subring S of T with $|S| = |R|$ such that $R \subseteq S \subseteq T$ and $p_i T \cap S = p_i S$ for all i.*

Proof. Let $R_0 = R$. We define R_i for $i = 1, 2, \ldots$ by induction. Assume R_{i-1} has been defined so that it is a PC-subring and $|R_{i-1}| = |R|$. Now use Lemma 2.6

to find a PC-subring R_i with $p_jT \cap R_{i-1} \subseteq p_jR_i$ for all $j = 1, 2\ldots$ and so that $|R_i| = |R_{i-1}| = |R|$. Let $S = \bigcup_{i=1}^{\infty} R_i$. By Lemma 2.3 we know that S is a PC-subring with $|S| = |R|$. Further, if $c \in p_iT \cap S$ for some i, there is an $n \in \mathbb{N}$ such that $c \in p_iT \cap R_n \subseteq p_iR_{n+1} \subseteq p_iS$. Therefore, $p_iT \cap S \subseteq p_iS$, and it follows that $p_iT \cap S = p_iS$ for all $i = 1, 2, \ldots$. □

If R is a PC-subring of T, then $Q_i \cap R \subseteq p_iT$ for all i. If we have the additional property given in Lemma 2.7 that $p_iT \cap R = p_iR$, then $Q_i \cap R = p_iR$. We also have in this case that, since Q_i is a prime ideal of T, p_iR is a prime ideal of R.

Eventually, we show that the completion of our final ring A is T. To do this, we use the following very useful proposition.

Proposition 2.8 ([6], Proposition 1). *If $(A, M \cap A)$ is a quasi-local subring of a complete local ring (T, M), the map $A \longrightarrow T/M^2$ is onto and $IT \cap A = I$ for every finitely generated ideal I of A, then A is Noetherian, and the natural homomorphism $\widehat{A} \longrightarrow T$ is an isomorphism.*

To control the formal fiber of A at (0) and the formal fibers at the ideals p_iA, as well as to ensure that the completion of A is T, we adjoin elements of T to a PC-subring of T so that they satisfy very specific transcendental properties. Lemma 2.9 allows us to do this. In many of the following results, to satisfy the cardinality condition of this lemma, we use condition (1) of PC-subrings.

Lemma 2.9 ([5], Lemma 3). *Let (T, M) be a local ring. Let $C \subset \operatorname{Spec} T$, let I be an ideal such that $I \not\subset P$ for every $P \in C$, and let D be a subset of T. Suppose $|C \times D| < |T/M|$. Then $I \not\subset \bigcup\{(P + r) \mid P \in C, r \in D\}$.*

Since our goal is for any prime ideal of T not contained in P or a Q_i to not be in the formal fiber of p_iA or (0), if J is a prime ideal of T such that $J \nsubseteq P$ and $J \nsubseteq Q$ for all $Q \in C$, then we want $A \cap J \neq (0)$ and $A \cap J \neq p_iA$ for every i. We also want the map $A \longrightarrow T/J$ to be onto. In the proof of Lemma 2.10, we show that, given a PC-subring R, we can construct a larger PC-subring S that contains an element from a specific coset of T/J and so that the property that $p_iT \cap S = p_iS$ for all i is satisfied.

Lemma 2.10. *Let (T, M), C, P, and $\{p_i\}$ be as in Definition 2.1 with the extra condition that for each i if $Q \in \operatorname{Ass}(T/p_iT)$, then $Q \subseteq Q_i$. Suppose further that $|T| = |T/M|$. Let $(R, R \cap M)$ be a PC-subring of T such that $p_iT \cap R = p_iR$ for each i, and let $u + J \in T/J$ where J is an ideal of T with $J \nsubseteq P$ and $J \nsubseteq Q$ for all $Q \in C$. Then there exists a PC-subring S of T meeting the following conditions*:

(1) $R \subseteq S \subseteq T$;
(2) $|S| = |R|$;
(3) *$u + J$ is in the image of the map $S \to T/J$*;
(4) *If $u \in J$, then $S \cap J \nsubseteq Q$, for all $Q \in C$*;
(5) *$p_iT \cap S = p_iS$ for all i.*

Proof. For each $P' \in \operatorname{Ass} T \cup \{P\}$, let $D_{(P')}$ be a full set of coset representatives of the cosets $t + P'$ that make $(u + t) + P'$ algebraic over R. For each $Q_i \in C$ let $D_{(Q_i)}$

be a full set of coset representatives of the cosets $t + Q_i \in T/Q_i$ with $t \in T$ that make $(u + t) + Q_i$ algebraic over $R/(R \cap Q_i)$. Let G be the set $C \cup \{P\} \cup \operatorname{Ass} T$, and note that if $Q \in G$, then $t + Q \neq t' + Q$ implies that $(u + t) + Q \neq (u + t') + Q$. Also, the algebraic closure of $R/(R \cap Q)$ in T/Q has cardinality at most $|R|$ and so $|D_{(Q)}| \leq |R|$. Let

$$D := \bigcup_{Q \in G} D_{(Q)}.$$

Then we have $|D| \leq |R| < |T| = |T/M|$.

We now claim that if P' is an associated prime ideal of T, then $J \nsubseteq P'$. Suppose that $b \in P'$. Then b is a zero-divisor, so $bc = 0$ for some $c \in T$ with $c \neq 0$. If $c \in p_1 T$, then we can write $c = p_1^n c'$ for some positive integer n and $c' \in T$ with $c' \notin p_1 T$. Now, $0 = bc = bp_1^n c'$, and as p_1 is not a zero-divisor, we have $bc' = 0$. Thus we may assume that $bc = 0$ for some $c \notin p_1 T$. Now, $b(c + p_1 T) = 0 + p_1 T$ in $T/p_1 T$ with $c + p_1 T \neq 0 + p_1 T$. It follows that $b \in Q$ for some $Q \in \operatorname{Ass}(T/p_1 T)$. By our hypothesis, we have that $b \in Q \subseteq Q_1$. Hence, $P' \subseteq Q_1$. Since $J \nsubseteq Q_1$, we have that $J \nsubseteq P'$ as desired.

Since $|G \times D| \leq |R| < |T| = |T/M|$, we can now employ Lemma 2.9 with $I = J$ to find an $x \in J$ such that $x \notin \bigcup\{r + Q \mid r \in D, Q \in G\}$. We claim that $S' = R[u + x]_{(R[u+x] \cap M)}$ is a PC-subring. It is clear that $|S'| = |R|$. Further, note that since $(u + x) + P'$ is transcendental over R for all $P' \in \operatorname{Ass} T \cup \{P\}$, we know if $f = r_n(u + x)^n + \cdots + r_0 \in R[u + x] \cap P'$ for some $P' \in \operatorname{Ass} T \cup \{P\}$, then $r_i \in R \cap P' = (0)$ for all i and so $f = 0$. We thus have $R[u + x] \cap P' = (0)$ for every $P' \in \operatorname{Ass} T \cup \{P\}$.

We now claim that $(Q_i \backslash p_i T)R[u + x] \cap R[u + x] = \{0\}$ for each i. First, suppose we have $f \in (Q_i \backslash p_i T)R[u + x] \cap R[u + x]$ for some i with $f \neq 0$. Then we have $f = r_n(u + x)^n + \cdots + r_0 = q(s_{n'}(u + x)^{n'} + \cdots + s_1(u + x) + s_0)$ for some $q \in Q_i \backslash p_i T$ and some $r_0, \ldots, r_n, s_0, \ldots, s_{n'} \in R$ with $r_k \neq 0$ for some $1 \leq k \leq n$. Let m be the largest integer such that $r_j \in (p_i T)^m$ for all $1 \leq j \leq n$ (this exists by the Krull Intersection theorem), and let m' be the largest integer such that $s_{j'} \in (p_i T)^{m'}$ for all $1 \leq j' \leq n'$. Then since $p_i T \cap R = p_i R$, we have $(p_i T)^m \cap R = p_i^m R$ (and similarly for m'), and we can write

$$f = p_i^m(r_n'(u + x)^n + \cdots + r_0') = qp_i^{m'}(s_n'(u + x)^{n'} + \cdots + s_0')$$

for some $r_0', \ldots, r_n', s_0', \ldots, s_n' \in R$.

By the maximality of m and m', we know there is an l such that $r_l' \notin p_i T$ and a j such that $s_j' \notin p_i T$. Since $(Q_i \backslash p_i T)R \cap R = \{0\}$, we have that $Q_i \cap R \subseteq p_i T$, and thus $r_l', s_j' \notin Q_i \cap R$. Since $(u + x) + Q_i$ is transcendental over $R/(R \cap Q_i)$ for all $i = 1, 2 \ldots$ we, therefore, know that

$$r_n'(u + x)^n + \cdots + r_1'(u + x) + r_0' \notin Q_i$$
$$s_{n'}'(u + x)^{n'} + \cdots + s_1'(u + x) + s_0' \notin Q_i.$$

Now suppose that $m \leq m'$. Since p_i is not a zero-divisor, we may cancel it on both sides of our equation to get

$$r_n'(u + x)^n + \cdots + r_1'(u + x) + r_0' = qp_i^{m'-m}(s_{n'}'(u + x)^{n'} + \cdots + s_1'(u + x) + s_0').$$

But the left-hand side is not in Q_i, while the right-hand side is clearly in Q_i, which is a contradiction.

On the other hand, suppose $m > m'$. Then canceling, we have

$$p_i^{m-m'}(r'_n(u+x)^n + \cdots + r'_1(u+x) + r'_0) = q(s'_{n'}(u+x)^{n'} + \cdots + s'_1(u+x) + s'_0).$$

The left-hand side is clearly in p_iT, but since $s'_{n'}(u+x)^{n'} + \cdots + s'_1(u+x) + s'_0$ is not in Q_i, it is not in any associated prime of p_iT, and so is not a zero-divisor of T/p_iT. Since $q \notin p_iT$, we have that the right-hand side is not in p_iT, which is a contradiction. Thus we have $((Q_i \backslash p_iT)R[u+x]) \cap R[u+x] = \{0\}$. We now use Lemma 2.2 to conclude that S' is a PC-subring of T.

We now employ Lemma 2.7 to find a PC-subring S with $S' \subseteq S \subseteq T$ and $|S| = |S'| = |R|$ such that $p_iT \cap S = p_iS$ for each i. Since $S' \subseteq S$, the image of S in T/J contains $u + x + J = u + J$. Furthermore, if $u \in J$, then $u + x \in J \cap S$, but since $(u+x) + Q_i$ is transcendental over $R/(R \cap Q_i)$ for each $i \in \{1, 2, \dots\}$, we have $u + x \notin Q_i$ so $J \cap S \nsubseteq Q_i$ for all i. $\square$

Remark 2.11. Note that from the proof of Lemma 2.10 we have that if R is a PC-subring of T and $x + Q_i \in T/Q_i$ is transcendental over $R/(Q_i \cap R)$ for every i, then $(Q_i \backslash p_iT)R[x] \cap R[x] = \{0\}$ for every i. We also have that if P' is a prime ideal of T with $R \cap P' = (0)$ and $x + P' \in T/P'$ is transcendental over R, then $R[x] \cap P' = (0)$.

Remark 2.12. By the proof of Lemma 2.10 if the condition $Q \in \mathrm{Ass}(T/p_iT)$ implies $Q \subseteq Q_i$ is satisfied, then Q_i contains all associated prime ideals of T.

Recall that to show that the completion of A is T, we use Proposition 2.8. In particular, we need $IT \cap A = I$ for all finitely generated ideals I of A. This is perhaps the most challenging part of the proof. Certainly, $I \subseteq IT \cap A$ trivially holds. Given a PC-subring R, we show that there is a larger PC-subring S satisfying $IT \cap S = I$ for all finitely generated ideals I of S. Theorem 2.13 is the first step in doing this. The next series of results is devoted to constructing this PC-subring S, and the result is finally given in Lemma 2.19.

Theorem 2.13. *Let (T, M), C, P, and $\{p_i\}$ be as in Definition 2.1 with the extra conditions that T is a UFD and $|T/M| = |T|$. Let $(R, R \cap M)$ be a PC-subring of T such that $p_iT \cap R = p_iR$, for each i. Suppose I is a finitely generated ideal of R, and let $c \in IT \cap R$. Then there exists a PC-subring S of T meeting the following conditions*:

(1) $R \subseteq S \subseteq T$;
(2) $|S| = |R|$;
(3) $c \in IS$;
(4) $p_iT \cap S = p_iS$ *for each i.*

The proof of Theorem 2.13 involves many steps. To make reading the proof easier, we break it up into several lemmas.

Lemma 2.14. *Theorem 2.13 holds if I is generated by one element.*

Proof. Suppose $I = aR$. If $a = 0$, then $c = 0$ so $S = R$ is the desired PC-subring. If $a \neq 0$, then $c = au$ for some $u \in T$. We claim that $S' = R[u]_{(R[u] \cap M)}$ is a PC-subring of T. First note that clearly $|S'| = |R| < |T|$. Suppose $f \in P \cap R[u]$. Then $f = r_n u^n + \cdots + r_1 u + r_0 \in P$, and $a^n f = r_n c^n + \cdots + r_1 c a^{n-1} + r_0 a^n \in P \cap R = (0)$. Since $a \in R$ and R contains no zero-divisors of T, we have $f = 0$. It follows that $R[u] \cap P = (0)$. A similar proof shows that $R[u]$ satisfies the second part of condition (2) of PC-subrings. Now suppose $f \in ((Q_i \backslash p_i T)R[u]) \cap R[u]$ for some i. Then $f = qg$, where $q \in Q_i \backslash p_i T$ and $g \in R[u]$. Since $c = au \in R$, from the argument above, we know there exists an m such that $a^m f \in R$ and $a^m g \in R$. Thus we have $a^m f \in (Q_i \backslash p_i T)R \cap R = \{0\}$, and since R contains no zero-divisors of T, we know $f = 0$. Therefore, $(Q_i \backslash p_i T)R[u] \cap R[u] = \{0\}$ for all i. By Lemma 2.2, we have that S' is a PC-subring. Now use Lemma 2.7, with $R = S'$ to construct the desired S. □

To prove Theorem 2.13 when I is generated by two elements, we first show that it suffices to prove Theorem 2.13 if the generators of I share no associated prime ideals. To do this, we use the following lemma.

Lemma 2.15 ([7], Lemma 4). *Suppose (T, M) is a local ring with $|T/M|$ infinite. Let $C_1, C_2 \subset \operatorname{Spec} T$, $u, w \in T$ such that $u \notin P$ for every $P \in C_1$ and $w \notin Q$ for every $Q \in C_2$. Also, suppose D_1 and D_2 are subsets of T. If $|C_1 \times D_1| < |T/M|$ and $|C_2 \times D_2| < |T/M|$, then we can find a unit $x \in T$ such that $ux \notin \bigcup\{P + r \mid P \in C_1,\ r \in D_1\}$ and $wx^{-1} \notin \bigcup\{Q + a \mid Q \in C_2, a \in D_2\}$.*

Lemma 2.16. *To prove Theorem 2.13, it suffices to prove it for the case $I = (y_1, y_2, \ldots, y_m)$ where $m \geq 2$ and $\operatorname{Ass}(T/y_1T) \cap \cdots \cap \operatorname{Ass}(T/y_mT) = \emptyset$.*

Proof. Note that by Lemma 2.14, Theorem 2.13 holds for $m = 1$. Now suppose $I = (y_1, \ldots, y_m)$ with $m \geq 2$ and $\operatorname{Ass}(T/y_1T) \cap \cdots \cap \operatorname{Ass}(T/y_mT) \neq \emptyset$. Since T is a UFD, we know there is a greatest common divisor of $y_1, y_2, \ldots y_m$, call it x. By our assumption, x is not a unit. Write $x = (r_1^{e_1} \cdots r_s^{e_s})w$ where each $r_j = p_i$ for some i, the e_i's are positive integers, and so that $w \notin p_i T$ for every i. If no p_i divides x, then set $s = 1$ and $r_1 = 1$. We claim that $w \notin P$ and that for every i, we have $w \notin Q_i$. Note that $y_1 = xz_1 = (r_1^{e_1} \cdots r_s^{e_s})wz_1$ for some $z_1 \in T$, so if $w \in P$, then $y_1 \in P \cap R = (0)$, a contradiction. Now suppose that for some i, we have $w \in Q_i$. Note that

$$
\begin{aligned}
y_1 &= (r_1^{e_1} \cdots r_s^{e_s})wz_1 \\
y_2 &= (r_1^{e_1} \cdots r_s^{e_s})wz_2 \\
&\vdots \\
y_m &= (r_1^{e_1} \cdots r_s^{e_s})wz_m
\end{aligned}
$$

for $z_1, z_2, \ldots, z_m \in T$. If p_i divides wz_k for all $k = 1, 2, \ldots, m$, then $p_i(r_1^{e_1} \cdots r_s^{e_s})$ divides y_j for all $j = 1, 2 \ldots, m$. Since x is a greatest common divisor for the y_j's, we have that $p_i(r_1^{e_1} \cdots r_s^{e_s})$ divides $x = (r_1^{e_1} \cdots r_s^{e_s})w$. Hence $p_i u = w$ for some $u \in T$. But this contradicts that $w \notin p_i T$. So we have that there is a j such that p_i does not divide wz_j. It follows that $y_j = (wz_j)(r_1^{e_1} \cdots r_s^{e_s}) \in (Q_i \backslash p_i T)R \cap R = \{0\}$, a contradiction. So we have shown that $w \notin P$ and that for every i, we have $w \notin Q_i$.

Now if P' is a prime ideal of T, let $D_{(P')}$ be a full set of coset representatives for those cosets $u + P' \in T/P'$ such that $u + P'$ is algebraic over $R/(R \cap P')$. Let $G = C \cup \{P\}$ and $D = \bigcup_{P' \in G} D_{(P')}$. Now use Lemma 2.15 to find a unit $t \in T$ satisfying

$$wt \notin \bigcup\{P' + r \mid P' \in G, r \in D\}.$$

Then we have that $wt + P'$ is transcendental over $R/(R \cap P')$ for all $P' \in G$. Now let $R' = R[wt]$. By Remark 2.11 and Lemma 2.2, we have that $S_0 = R'_{(R' \cap M)}$ is a PC-subring. Note that since T is an integral domain, the second part of condition (2) of PC-subrings is satisfied automatically. We also have that $xt = (r_1^{e_1} \cdots r_s^{e_s})(wt) \in S_0$.

Now, $y_1 \in (xt)T \cap S_0$, so use the proof of Lemma 2.14 to construct a PC-subring S_1 so that $S_0 \subseteq S_1 \subseteq T$, $|S_1| = |S_0| = |R|$, and $y_1 \in (xt)S_1$. Now, $y_2 \in (xt)T \cap S_1$ so repeat this construction to find a PC-subring S_2 so that $S_1 \subseteq S_2 \subseteq T$, $|S_2| = |R|$, and $y_2 \in (xt)S_2$. Keep going, so that for every j with $1 \leq j \leq m$, we have $S_{j-1} \subseteq S_j \subseteq T$, $|S_j| = |R|$, and $y_j \in (xt)S_j$. Note that $c \in (xt)T \cap S_m$, so we can do the construction one more time to construct a PC-subring S'' satisfying $R \subseteq S'' \subseteq T$, $|S''| = |R|$, $c \in (xt)S''$, and $y_j \in (xt)S''$ for all j satisfying $1 \leq j \leq m$. Use Lemma 2.7 to construct a PC-subring S^* satisfying the above properties and that $p_i T \cap S^* = p_i S^*$ for each i. Let $c' = c/(xt)$ and $y'_j = y_j/(xt)$ for $j = 1, 2 \ldots, m$. Then $c' \in (y'_1, \ldots, y'_m)T \cap S^*$ and $\operatorname{Ass}(T/y'_1 T) \cap \cdots \cap \operatorname{Ass}(T/y'_m T) = \emptyset$. So we can use our assumption that Theorem 2.13 holds in this case to find a PC-subring S such that $S^* \subseteq S \subseteq T$, $|S| = |S^*|$, $c' \in (y'_1, \ldots, y'_m)S$, and $p_i T \cap S = p_i S$ for every i. It follows that $R \subseteq S \subseteq T$, $|S| = |R|$, and $c = (xt)c' \in ((xt)y'_1, \ldots, (xt)y'_m)S = (y_1, \ldots, y_m)S$. □

Lemma 2.17. *Theorem* 2.13 *holds if I is generated by two elements.*

Proof. We now assume $I = (y_1, y_2)$. By Lemma 2.16, we may reduce to the case $\operatorname{Ass}(T/y_1 T) \cap \operatorname{Ass}(T/y_2 T) = \emptyset$. Our proof follows closely the proof of Lemma 4 in [5]. Now $c = y_1 t_1 + y_2 t_2$ for some $t_1, t_2 \in T$. We write $c = (t_1 + ty_2)y_1 + (t_2 - ty_1)y_2$, where we will choose $t \in T$ in a strategic way later. Let $x_1 = t_1 + ty_2$ and $x_2 = t_2 - ty_1$. Then we have $c = x_1 y_1 + x_2 y_2$. Let $R' = R[x_1, y_2^{-1}] \cap R[x_2, y_1^{-1}]$ and note that since $x_1 = (c - x_2 y_2)/y_1$ and $x_2 = (c - x_1 y_1)/y_2$, we have $x_1, x_2 \in R'$ and so $c \in (y_1, y_2)R'$.

We now show that $R' \subseteq T$. Note that $R' \subseteq T[y_1^{-1}] \cap T[y_2^{-1}]$. Let $f \in T[y_1^{-1}] \cap T[y_2^{-1}]$. Then $f = t/y_2^n$ for some $t \in T$ and some nonnegative integer n. We also have that $f = s/y_1^m$ for some $s \in T$ and some nonnegative integer m. It follows that $sy_2^n = ty_1^m \in T$, and since y_1 and y_2 are relatively prime in T, we have that y_1^m must divide s in T. Hence, $f \in T$, and so $R' \subseteq T[y_1^{-1}] \cap T[y_2^{-1}] = T$.

We will now work to define t. Let $G = C \cup \{P\}$. If $y_1, y_2 \in Q_i$ for some i, then $y_1, y_2 \in Q_i \cap R \subseteq p_i T$. This contradicts that $\operatorname{Ass}(T/y_1 T) \cap \operatorname{Ass}(T/y_2 T) = \emptyset$. So for every $Q \in G$, we have y_1 or y_2 is not in Q. Now, let $Q \in G$, and suppose $y_1 \notin Q$. Define $D_{(Q)}$ to be a full set of coset representatives of the cosets $t + Q \in T/Q$ that make $x_2 + Q = (t_2 - ty_1) + Q$ algebraic over $R/(Q \cap R)$. Suppose $t + Q \neq t' + Q$. Then if $(t_2 - ty_1) + Q = (t_2 - t'y_1) + Q$, we have $y_1(t - t') \in Q$. As $y_1 \notin Q$, we have $t + Q = t' + Q$, a contradiction. It follows that different choices of the coset $t + Q$ will give us different cosets $x_2 + Q$. If $y_1 \in Q$, then $y_2 \notin Q$. In this case, let $D_{(Q)}$ be a full set of coset representatives of the cosets $t + Q \in T/Q$ that make $x_1 + Q = (t_1 + ty_2) + Q$ algebraic over $R/(Q \cap R)$. Using the same argument as above, we have that different choices of the coset $t + Q$ will give different cosets $x_1 + Q$. Let $D = \bigcup_{Q \in G} D_{(Q)}$. Then we have $|D \times G| < |T| = |T/M|$. Now use Lemma 2.9 with $I = M$ to find $t \in T$ such that $t \notin \cup\{(Q + r) \mid Q \in G, r \in D\}$. It follows that for this t, if $Q \in G$ with $y_1 \notin Q$, then $x_2 + Q$ is transcendental over $R/(Q \cap R)$ and otherwise, $y_2 \notin Q$ and $x_1 + Q$ is transcendental over $R/(Q \cap R)$.

Clearly, $|R'| = |R|$. We now show that R' satisfies conditions (2) and (3) for PC-subrings. Let $f \in R' \cap P$. Then multiplying through by a high enough power of y_1, we get $y_1^s f \in P \cap R[x_2]$. But by the way we chose t, $x_2 + P$ is transcendental over R, so $P \cap R[x_2] = (0)$. It follows that $f = 0$. Now suppose that for some i, we have $g \in (Q_i \backslash p_i T)R' \cap R'$. We know that $y_1 \notin Q_i$ or $y_2 \notin Q_i$. Without loss of generality, suppose $y_1 \notin Q_i$. Then $x_2 + Q_i$ is transcendental over $R/(R \cap Q_i)$. By the argument in the proof of Lemma 2.10, we have that $(Q_i \backslash p_i T)R[x_2] \cap R[x_2] = \{0\}$. Now, $g = qz$ for some $q \in (Q_i \backslash p_i T)$ and $z \in R'$. Multiplying through by a high enough power of y_1, we get $y_1^s g = q(y_1^s z)$, where $y_1^s g \in R[x_2]$ and $y_1^s z \in R[x_2]$. So we have $y_1^s g \in (Q_i \backslash p_i T)R[x_2] \cap R[x_2] = \{0\}$. It follows that $g = 0$. Now by Lemma 2.2, we have that $S' = R'_{(M \cap R')}$ is a PC-subring. Since $c \in (y_1, y_2)R'$, we have $c \in (y_1, y_2)S'$. Now use Lemma 2.7 to get the desired PC-subring S. □

We are now ready to prove Theorem 2.13. We will induct on the number of generators of I.

Proof of Theorem 2.13. Let $I = (y_1, \ldots, y_m)$. We will induct on m. If $m = 1$, then by Lemma 2.14 the theorem holds. Likewise, if $m = 2$, then the theorem holds by Lemma 2.17. So suppose $m > 2$, and assume the theorem holds for all ideals with $m - 1$ generators. Our proof follows the proof of Lemma 4 in [5] closely. We will construct a PC-subring S' so that $R \subseteq S' \subseteq T$, $|S'| = |R|$, there is an element $c^* \in S'$ and an ideal J of S' generated by $m - 1$ elements and $c^* \in JT$. S' will also satisfy the condition that $S' \cap p_i T = p_i S'$ for all $i \in \{1, 2, \ldots\}$. Then by our induction assumption, there is a PC-subring S satisfying $S' \subseteq S \subseteq T$, $|S| = |S'|$, $p_i T \cap S = p_i S$ for each i, and $c^* \in JS$. We will then show that $c \in IS$, proving the theorem.

We now work to construct S'. Let $J = (y_1, y_2, \ldots, y_{m-1})R$. Since $c \in IT$, we can write $c = y_1 t_1 + \cdots + y_m t_m$ for some $t_j \in T$. We first deal with the case where there is no Q_i satisfying $\{y_1, y_2, \ldots, y_{m-1}\} \subseteq Q_i$. Now let $v = t_m + u_1 y_1 + \cdots + u_{m-1} y_{m-1}$, where we will choose the $u_j \in T$ in a strategic way later. Let $R' = R[v]$ and $c^* = c - y_m v$. Then

$$c^* = (y_1 t_1 + \cdots + y_m t_m) - y_m(t_m + u_1 y_1 + \cdots + u_{m-1} y_{m-1}),$$

and so we have that $c^* \in JT$. To choose the u_j's, let $G = C \cup \{P\}$. Suppose $Q \in G$ with $y_1 \notin Q$. Then let $D_{(Q)}$ be a full set of coset representatives for the cosets $z + Q$ that make $(t_m + zy_1) + Q$ algebraic over $R/(R \cap Q)$. Let $D = \bigcup_{Q \in G, y_1 \notin Q} D_{(Q)}$. Use Lemma 2.9 to find u_1 so that $(t_m + u_1 y_1) + Q$ is transcendental over $R/(Q \cap R)$ for all $Q \in G$ with $y_1 \notin Q$. Continue this process to get a set $\{u_1, \ldots, u_{m-1}\}$ so that $(t_m + u_1 y_1 + \cdots + u_{m-1} y_{m-1}) + Q = v + Q$ is transcendental over $R/(R \cap Q)$ for all $Q \in G$. Since there is no Q_i satisfying $\{y_1, y_2, \ldots, y_{m-1}\} \subseteq Q_i$ such a set $\{u_1, u_2, \ldots, u_{m-1}\}$ exists. By Remark 2.11 and Lemma 2.2, we have that $S'' = R'_{(R' \cap M)}$ is a PC-subring of T. Now use Lemma 2.7 to get the desired PC-subring S'. Use induction as explained in the previous paragraph to get the PC-subring S. We are left to show that $c \in IS$. But this is clear since $c = c^* + y_m v$, $c^* \in JS$ and $v \in S'$.

On the other hand, suppose that $\{y_1, \ldots, y_{m-1}\} \subseteq Q_i$ for some i. If this were true for infinitely many i's, then $y_1 \in Q_i \cap R \subseteq p_i T$ for infinitely many i's. But since $p_i \in Q_j$ if and only if $i = j$, this implies that y_1 is in infinitely many height one prime ideals of T, a contradiction. So we have that $\{y_1, \ldots, y_{m-1}\} \subseteq Q_i$ for finitely many i's. For such an i, we have $y_j \in Q_i \cap R \subseteq p_i T \cap R = p_i R$, and so we can write $y_j = p_i r'_j$ for all $j = 1, 2 \ldots, m-1$, where $r'_j \in R$. If $\{r'_1, \ldots, r'_{m-1}\} \subseteq Q_i$, repeat this until we get that $y_j = p_i^k s_j$ for all $j = 1, 2 \ldots, m-1$ where $s_j \in R$ and $\{s_1, \ldots, s_{m-1}\} \not\subseteq Q_i$. If $\{s_1, \ldots, s_{m-1}\} \subseteq Q_l$, then repeat the above procedure for p_l. Eventually, we get that $y_j = dr_j$ for every $j = 1, 2, \ldots, m-1$ where d is a (finite) product of the p_i's, $r_j \in R$ and $\{r_1, \ldots, r_{m-1}\} \not\subseteq Q_i$ for all i. Now let $w = t_1 r_1 + \cdots + t_{m-1} r_{m-1}$. Then $c = t_m y_m + (t_1 y_1 + \cdots + t_{m-1} y_{m-1}) = t_m y_m + d(t_1 r_1 + \cdots + t_{m-1} r_{m-1}) = t_m y_m + dw$. So we have that $c \in (y_m, d)T \cap R$. Use Lemma 2.17 to find a PC-subring R' of T such that $R \subseteq R' \subseteq T$, $|R'| = |R|$, $p_i T \cap R' = p_i R'$ for all i, and $c \in (y_m, d)R'$. Write $c = x_1 y_m + x_2 d$ with $x_1, x_2 \in R'$. Note that x_1 and x_2 come from Lemma 2.17 where, since $c = t_m y_m + dw$, w takes the role of t_2, d the role of y_2, y_m the role of y_1, and t_m the role of t_1 in Lemma 2.17 so that, in particular, $x_2 = w - ty_m$ for some $t \in T$. By the way, w is defined, we have that $x_2 = w - ty_m \in (r_1, r_2, \ldots, r_{m-1}, y_m)T \cap R'$. Now let $I^* = (r_1, \ldots, r_{m-1}, y_m)R'$ and $J^* = (r_1, \ldots, r_{m-1})R'$. Then $\{r_1, \ldots, r_{m-1}\} \not\subseteq Q_i$ for all i. So we can use the result from the previous paragraph with $c = x_2$ to construct a PC-subring S' so that $R' \subseteq S' \subseteq T$, $|S'| = |R'|$, and an element $c^* = x_2 - y_m v \in S'$ with $c^* \in J^* T$ and $v \in S'$. Also, we have that $S' \cap p_i T = p_i S'$ for all i. Now we use our induction assumption as explained in the first paragraph of this proof to get S. This gives us that $c^* = x_2 - y_m v \in J^* S$. We have left to show that $c \in IS$. We have that $c = x_1 y_m + x_2 d = x_1 y_m + (c^* + y_m v)d = c^* d + (x_1 + vd)y_m$. As $c^* \in J^* S$, we have that $c^* d \in (y_1, \ldots, y_{m-1})S$. We also have that $x_1 + vd \in S$ and so $c \in IS$ as desired. □

Theorem 2.18. *Let (T, M), C, P, and $\{p_i\}$ be as in Definition* 2.1 *with the set C containing $k < \infty$ elements. Then Theorem* 2.13 *holds even if we remove the condition that T is a UFD.*

Proof. We induct on the number of generators of I. Suppose I is generated by m elements. If $m = 1$, then we can use Lemma 2.14 since we did not use in that proof that T was a UFD. So for the rest of the proof, we assume $m > 1$.

Let $I = (y_1, y_2, \ldots, y_m)$. First suppose that $y_j \in p_1 T$ for all $j = 1, 2, \ldots, m$. Then since $p_1 T \cap R = p_1 R$, we can write $y_j = p_1 y'_j$ for each j to obtain $I = p_1 I'$, where $I' = (y'_1, y'_2, \ldots, y'_m)$. Now $c \in p_1 T \cap R = p_1 R$, so we have $c/p_1 \in I'$. We continue this process (which must terminate at some point since T is Noetherian) until we have an ideal $J_1 = (z_1, z_2, \ldots, z_m)$ satisfying $z_j \notin p_1 T$ for some j and such that there is a $d_1 \in R$ so that $d_1 J_1 = I$ and $c/d_1 \in J_1$. Repeat this process with the y_j's replaced by the z_j's and p_1 replaced by p_2 to find an ideal $J_2 = (w_1, w_2, \ldots, w_m)$ satisfying $w_j \notin p_1 T$, $w_l \notin p_2 T$ for some j and l and such that there is a d_2 so that $d_2 J_2 = I$ and $c/d_2 \in J_2$. Continue until we get an ideal $J_k = (u_1, u_2, \ldots, u_m)$ satisfying the condition that given $p_i T$, there is a j such that $u_j \notin p_i T$ and so that there is a d_k with $d_k J_k = I$ and $c/d_k \in J_k$. If there exists a PC-subring S such that $c/d_k \in J_k S$, then $c \in d_k J_k S = IS$. Thus it suffices to prove the theorem assuming there is no p_i with $y_j \in p_iT$ for all $j = 1, 2, \dots, m$. Note that since $p_iT \cap R = p_iR$, this is the same as assuming there is no p_i with $y_j \in p_iR$ for all $j = 1, 2, \dots, m$. We assume this for the rest of the proof.

We now show that we can find a set $\{z_1, z_2, \dots, z_m\} \subset R$ such that $I = (z_1, z_2, \dots, z_m)$ and $z_1 \notin p_iR$ for all $i = 1, 2, \dots, k$. If $y_1 \notin p_iR$ for all $i = 1, 2, \dots, k$, then choose $z_i = y_i$, and we are done. Now, set $x_1 = y_1$, and define $\pi(x_i) = \{p_j \mid x_i \notin p_jR\}$. Let $x_l = x_{l-1} + r_ly_l$, where $r_l = \prod_{p_i \in \pi(x_{l-1})} p_i$. We claim that $I = (x_m, y_2, \dots, y_n)$ and $x_m \notin p_iR$ for all $i = 1, 2, \dots, k$. The first statement is clear, since $x_m = y_1 + r_2y_2 + \cdots + r_my_m \in I$ and $y_1 = x_m - r_2y_2 - \cdots - r_my_m \in (x_m, y_2, \dots, y_n)$. To prove the second statement, fix i, and choose the smallest j such that $y_j \notin p_iR$. We know such a j exists from the previous paragraph. We claim $x_j \notin p_iR$. If $j = 1$, then $x_1 \notin p_iR$. Now suppose $j > 1$. Then by the choice of j, we have that $y_\ell \in p_iR$ for all $\ell < j$, so $x_{j-1} \in p_iR$. Now $x_j = x_{j-1} + r_jy_j$. We know that $y_j \notin p_iR$ and by construction of r_j, we have $r_j \notin p_iR$. Since $Q_i \cap R = p_iR$, we know p_iR is a prime ideal of R. It follows that $r_jy_j \notin p_iR$ and so $x_j \notin p_iR$. Now, $x_{j+1} = x_j + r_{j+1}y_{j+1}$ and $r_{j+1} \in p_iR$. It follows that $x_{j+1} \notin p_iR$. Continue until we get that $x_m \notin p_iR$. Choosing $z_1 = x_m$ and $z_i = y_i$ for $i = 2, 3, \dots, m$ we get the desired set $\{z_1, z_2, \dots, z_m\}$.

By the above paragraph, we can assume that $I = (y_1, y_2, \dots, y_m)$ and $y_2 \notin p_iR$ for all $i = 1, 2, \dots, k$. Note that this implies $y_2 \notin Q_i$ for every i. Since $c \in IT \cap R$, we can write $c = t_1y_1 + \cdots + t_my_m$ for $t_i \in T$. Set $x_1 = t_1 + y_2t$ and $x_2 = t_1 - y_2t$ for some $t \in T$ which we will choose later. Now we have $c = x_1y_1 + x_2y_2 + t_3y_3 + \cdots + t_ny_n$. Our goal is to adjoin x_1 to our subring R without disturbing the PC-subring properties.

Let $G = \operatorname{Ass} T \cup \{P\} \cup C$. For each $Q \in G$, let $D_{(Q)}$ be the full set of coset representatives of the cosets $t + Q$ that make $t_1 + y_2t + Q$ algebraic over $R/(R \cap Q)$. Let $D = \bigcup_{Q \in G} D_{(Q)}$. Note if $(t_1 + y_2t) + Q = (t_1 + y_2t') + Q$, then $y_2(t - t') \in Q$, and since $y_2 \notin Q$, we have $t + Q = t' + Q$. Thus for all $Q \in G$, we have that if $t + Q \neq t' + Q$, then $(t_1 + y_2t) + Q \neq (t_1 + y_2t) + Q$. This argument shows that $|D| = |R| < |T| = |T/M|$.

We now use Lemma 2.9 with $I = T$ to find an element $t \in T$ such that $t \notin \bigcup\{r + P \mid r \in D, P \in G\}$. Thus we have that $x_1 + Q = t_1 + y_2t + Q$ is transcendental over $R/(R \cap Q)$ for all $Q \in G$. So by the proof of Lemma 2.10 we have $S' = R[x_1]_{(R[x_1] \cap M)}$ is a PC-subring. Now use Lemma 2.7 to get a PC-subring so that $p_iT \cap S'' = p_iS''$ for all i. Let $J = (y_2, \dots, y_m)S''$ and $c^* = c - y_1x_1$. Clearly, $c^* \in JT \cap S''$, and so by induction, we can find a PC-subring S of T such that $S'' \subseteq S \subseteq T$ and $c^* \in JS$. So $c^* = s_2y_2 + \cdots + s_my_m$ for some $s_i \in S$. Therefore, $c = x_1y_1 + s_2y_2 + \cdots + s_my_m \in IS$ and the result follows. □

Lemma 2.19. *Let (T, M), C, P, and $\{p_i\}$ be as in Definition 2.1 with the extra condition that for each i if $Q \in \operatorname{Ass}(T/p_iT)$, we have $Q \subseteq Q_i$. Suppose further that T is a UFD and $|T| = |T/M|$. Let $(R, R \cap M)$ be a PC-subring of T such that $p_iT \cap R = p_iR$ for every i, let J be an ideal of T with $J \nsubseteq P$ and $J \nsubseteq Q$ for all $Q \in C$, and let $u + J \in T/J$. Then there exists a PC-subring S of T such that*:

(1) $R \subseteq S \subseteq T$;
(2) $|S| = |R|$;
(3) *$u + J$ is in the image of the map $S \to T/J$*;

(4) *If* $u \in J$, *then* $S \cap J \nsubseteq Q$ *for all* $Q \in C$;
(5) *For every finitely generated ideal* I *of* S, *we have* $IT \cap S = I$.

Proof. We first apply Lemma 2.10 to find a PC-subring R' of T satisfying conditions 1, 2, 3, and 4 and such that $p_iT \cap R' = p_iR'$ for each i. We will now construct the desired S such that S satisfies conditions 2 and 5 and $R' \subseteq S \subseteq T$ which will ensure that the first, third, and fourth conditions of the lemma hold true. Let $\Omega = \{(I, c) \mid I$ is a finitely generated ideal of R' and $c \in IT \cap R'\}$. Letting $I = R'$, we see that $|\Omega| \geq |R'|$. Since R' is infinite, the number of finitely generated ideals of R' is $|R'|$, and therefore $|R'| \geq |\Omega|$, giving us the equality $|R'| = |\Omega|$. Well order Ω so that it does not have a maximal element and let 0 denote its first element. We will now inductively define a family of PC-subrings of T, one for each element of Ω. Let $R_0 = R'$, and let $\alpha \in \Omega$. Assume that R_β has been defined for all $\beta < \alpha$ and that $p_iT \cap R_\beta = p_iR_\beta$ and $|R_\beta| = |R|$ hold for all $\beta < \alpha$. If $\gamma(\alpha) < \alpha$ and $\gamma(\alpha) = (I, c)$, then define R_α to be the PC-subring obtained from Theorem 2.13 so that $c \in IR_\alpha$. Note that clearly $p_iT \cap R_\alpha = p_iR_\alpha$ and $|R_\alpha| = |R_{\gamma(\alpha)}| = |R|$. If on the other hand $\gamma(\alpha) = \alpha$, define $R_\alpha = \bigcup_{\beta<\alpha} R_\beta$. By Lemma 2.3 R_α is a PC-subring with $|R_\alpha| = |R|$. Furthermore, if $t \in p_iT \cap R_\alpha$ for some i, then $t \in R_\beta$ for some $\beta < \alpha$, and so $t \in p_iT \cap R_\beta = p_iR_\beta \subseteq p_iR_\alpha$. Thus $p_iT \cap R_\alpha = p_iR_\alpha$.

Now let $R_1 = \bigcup_{\alpha\in\Omega} R_\alpha$. We see from Lemma 2.3 that R_1 is a PC-subring and $|R_1| = |R_0| = |R|$. Also, since we know by induction that $p_iT \cap R_\alpha = p_iR_\alpha$ for all $\alpha \in \Omega$ we see by the same argument made at the end of the last paragraph that $p_iT \cap R_1 = p_iR_1$ for all i. Furthermore, notice that if I is a finitely generated ideal of R_0 and $c \in IT \cap R_0$, then $(I, c) = \gamma(\alpha)$ for some $\alpha \in \Omega$ with $\gamma(\alpha) < \alpha$. It follows from the construction that $c \in IR_\alpha \subseteq IR_1$. Thus $IT \cap R_0 \subseteq IR_1$ for every finitely generated ideal I of R_0.

Following this same pattern, build a PC-subring R_2 of T with $|R_2| = |R_1| = |R|$ and $p_iT \cap R_2 = p_iR_2$ for all i and such that $R_1 \subseteq R_2 \subseteq T$ and $IT \cap R_1 \subseteq IR_2$ for every finitely generated ideal I of R_1. Continue to form a chain $R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots$ of PC-subrings of T such that $IT \cap R_n \subseteq IR_{n+1}$ for every finitely generated ideal I of R_n and $|R_j| = |R_0|$ for all j.

We now claim that $S = \bigcup_{i=1}^{\infty} R_i$ is the desired PC-subring. To see this, first note $R \subseteq S \subseteq T$ and that we know from Lemma 2.3 that S is indeed a PC-subring and $|S| = |R|$. Now set $I = (y_1, y_2, \ldots, y_k)S$, and let $c \in IT \cap S$. Then there exists an $N \in \mathbb{N}$ such that $c, y_1, \ldots, y_k \in R_N$. Thus $c \in (y_1, \ldots, y_k)T \cap R_N \subseteq (y_1, \ldots, y_k)R_{N+1} \subseteq IS$. From this it follows that $IT \cap S = I$, so the fifth condition of the statement of the lemma holds. $\square$

We are ready to prove the version of our main result, where we do not require that A be excellent.

Theorem 2.20. *Let* (T, M), C, P, *and* $\{p_i\}$ *be as in Definition* 2.1 *with the extra condition that for each* i *if* $Q \in \mathrm{Ass}(T/p_iT)$, *we have* $Q \subseteq Q_i$. *Suppose further that* T *is a UFD and* $|T| = |T/M|$. *Suppose that a PC-subring of* T, $(R, R \cap M)$, *exists. Then there exists a local UFD* $A \subseteq T$ *such that*:

(1) $p_i \in A$ *for all* i;
(2) $\widehat{A} = T$;

(3) *$A \cap P = (0)$ and if J is a prime ideal of T with $J \cap A = (0)$, then $J \subseteq P$ or $J \subseteq Q_i$ for some i;*
(4) *For each i, p_iA is a prime ideal in A and has a local formal fiber with maximal ideal Q_i;*
(5) *If J is an ideal of T satisfying $J \not\subseteq P$ and $J \not\subseteq Q_i$ for all $i \in \{1, 2, \dots\}$, then the map $A \to T/J$ is onto and $J \cap A \not\subseteq Q$ for all $Q \in C$.*

Proof. Let $G = C \cup \{P\}$ and $\Omega = \{u + J \in T/J \mid J$ is an ideal of T with $J \not\subseteq Q$ for all $Q \in G\}$. Since T is infinite and Noetherian, $|\{J$ is an ideal of T with $J \not\subseteq Q$ for all $Q \in G\}| \leq |T|$. Also, if J is an ideal of T, then $|T/J| \leq |T|$. It follows that $|\Omega| \leq |T|$. Well order Ω so that each element has fewer than $|\Omega|$ predecessors. Let 0 denote the first element of Ω. Apply Lemma 2.7 to find a PC-subring R'_0 with $R \subseteq R'_0$ such that for each i, $p_iT \cap R'_0 = p_iR'_0$ and $|R'_0| = |R|$. Next apply Lemma 2.19 with $J = M$ to find a PC-subring R_0 with $R'_0 \subseteq R_0$ such that $IT \cap R_0 = I$ for every finitely generated ideal I of R_0 and $|R_0| = |R'_0| = |R|$.

Starting with R_0, recursively define a family of PC-subrings as follows. Let $\alpha \in \Omega$, and assume that R_β has already been defined to be a PC-subring for all $\beta < \alpha$ with $IT \cap R_\beta = IR_\beta$ for every finitely generated ideal I of R_β and $|R_\beta| = \sup_{\delta<\beta}\{|\{\omega \in \Omega \mid \omega < \beta\}|, |R_\delta|\}$. Then $\gamma(\alpha) = u + J$ for some ideal J of T with $J \not\subseteq Q$ for every $Q \in G$. If $\gamma(\alpha) < \alpha$, use Lemma 2.19 to obtain a PC-subring R_α with $|R_\alpha| = |R_{\gamma(\alpha)}|$ such that $R_{\gamma(\alpha)} \subseteq R_\alpha \subseteq T$, $u + J$ is in the image of the map $R_\alpha \to T/J$ and $IT \cap R_\alpha = I$ for every finitely generated ideal I of R_α. Moreover, this gives us that $R_\alpha \cap J \not\subseteq Q$ for every $Q \in C$ if $u \in J$. Also, since $|R_\alpha| = |R_{\gamma(\alpha)}|$, and we have that $|R_\alpha| = \sup_{\delta<\alpha}\{|\{\omega \in \Omega \mid \omega < \alpha\}|, |R_\delta|\}$.

If $\gamma(\alpha) = \alpha$, define $R_\alpha = \bigcup_{\beta<\alpha} R_\beta$. Then by Lemma 2.3, we see that R_α is a PC-subring of T. Moreover, $|R_\alpha| = \sup_{\delta<\alpha}\{|\{\omega \in \Omega \mid \omega < \alpha\}|, |R_\delta|\}$. Now let $I = (y_1, \dots, y_k)$ be a finitely generated ideal of R_α, and let $c \in IT \cap R_\alpha$. Then $\{c, y_1, \dots, y_k\} \subseteq R_\beta$ for some $\beta < \alpha$. By the inductive hypothesis, $(y_1, \dots, y_k)T \cap R_\beta = (y_1, \dots, y_k)R_\beta$. As $c \in (y_1, \dots, y_k)T \cap R_\beta$, we have that $c \in (y_1, \dots, y_k)R_\beta \subseteq I$. Hence $IT \cap R_\alpha = I$.

We now know by induction that for each $\alpha \in \Omega$, R_α is a PC-subring with $|R_\alpha| = \sup_{\delta<\alpha}\{|\{\omega \in \Omega \mid \omega < \alpha\}|, |R_\delta|\}$ and $IT \cap R_\alpha = I$ for all finitely generated ideals I of R_α. We claim that $A = \bigcup_{\alpha\in\Omega} R_\alpha$ is the desired domain.

First note that by construction, condition (5) of the lemma is satisfied. We now show that the completion of A is T. Note that as Q is nonmaximal in T for all $Q \in G$, we have that $M^2 \not\subseteq Q$ for all $Q \in G$. Thus, by the construction, the map $A \to T/M^2$ is onto. Furthermore, by an argument identical to the one used to show that $IT \cap R_\alpha = I$ for all finitely generated ideals I of R_α in the case $\gamma(\alpha) = \alpha$, we know $I'T \cap A = I'$ for all finitely generated ideals I' of A. It follows from Proposition 2.8 that A is Noetherian and $\widehat{A} = T$. Since the completion of A is a UFD, A must also be a UFD.

Since each R_α is a PC-subring, we have that $A \cap P = (0)$. If J is a prime ideal of T with $J \not\subseteq P$ and $J \not\subseteq Q_i$ for all i, then by condition (5) $A \cap J \not\subseteq Q_i$. It follows that $A \cap J \neq (0)$. So, (3) holds for the lemma.

Now we show that the formal fiber of p_iA is local with maximal ideal Q_i. Since each R_α is a PC-subring, by the argument in Lemma 2.3, we know that $((Q_i \backslash p_iT)A) \cap A = \{0\}$ for all i and so in particular $(Q_i \backslash p_iT) \cap A = \emptyset$ for all i. Thus $Q_i \cap A = p_iT \cap A = p_iA$ for each i, and so p_iA is a prime ideal of A, and Q_i is in its formal fiber. Let J be a prime ideal of T with $J \not\subseteq Q_i$. We shall show that $J \cap A \neq p_i A$. If $J \subseteq P$, then $J \cap A = (0)$. If $J \subseteq Q_j$ for some $j \neq i$, then $J \cap A = p_j A \neq p_i A$. So suppose $J \not\subseteq Q$ for all $Q \in G$. Then by condition (5), $J \cap A \not\subseteq Q_i$. It follows that $J \cap A \neq p_i A$. Hence the formal fiber of $p_i A$ is local with maximal ideal Q_i. □

Note that the only reason we need to assume T is a UFD is to invoke Theorem 2.13. Using Theorem 2.18 in place of Theorem 2.13, we have the following theorem, which is a generalization of Theorem 2.13 in [2].

Theorem 2.21. *Let (T, M), C, P, and $\{p_i\}$ be as in Definition 2.1 with the extra condition that for each i if $Q \in \operatorname{Ass}(T/p_i T)$, we have $Q \subseteq Q_i$. Suppose further that C is a finite set and $|T| = |T/M|$. Suppose that a PC-subring of T, $(R, R \cap M)$, exists. Then there exists a local domain $A \subseteq T$ such that*:

(1) *$p_i \in A$ for all i;*
(2) *$\widehat{A} = T$;*
(3) *$A \cap P = (0)$, and if J is a prime ideal of T with $J \cap A = (0)$, then $J \subseteq P$ or $J \subseteq Q_i$ for some i;*
(4) *For each i, $p_i A$ is a prime ideal in A and has a local formal fiber with maximal ideal Q_i;*
(5) *If J is an ideal of T satisfying $J \not\subseteq P$ and $J \not\subseteq Q_i$ for all $i \in \{1, 2, \dots\}$, then the map $A \to T/J$ is onto and $J \cap A \not\subseteq Q$ for all $Q \in C$.*

Proof. Proceed in the same manner as the proof of Theorem 2.20 using Theorem 2.18 in place of Theorem 2.13. Note that by Remark 2.12, the set G used in the proof of Theorem 2.20 works even if T is not an integral domain. □

In light of Theorems 2.20 and 2.21, it is important to show that, for many complete local rings, a PC-subring indeed does exist. The next two lemmas are dedicated to showing that PC-subrings exist in certain cases.

Lemma 2.22. *Let (T, M) be a complete local ring such that $|T/M| = |T|$, and let P be a nonmaximal prime ideal of T. Let $\{q_1, q_2, \dots\}$ be a (nonempty) countable or finite set of nonzero prime elements of T. For each q_i, let Q_i be a nonmaximal prime ideal of T satisfying the property that $q_j \in Q_i$ if and only if $i = j$. Suppose also that $P \cap \Pi = (0)$, $Q \cap \Pi = (0)$ for all $Q \in \operatorname{Ass} T$, and for all i, $Q_i \cap \Pi = (0)$ where Π is the prime subring of T. Then there exists a PC-subring of T with respect to a set $\{p_1, p_2, \dots\}$, where $C = \{Q_1, Q_2, \dots\}$ and p_i is an associate of q_i for every i.*

Proof. Let $R_0 = \Pi_{(M \cap \Pi)}$. Then it is easy to see R_0 satisfies all conditions for being a PC-subring with respect to the set $\{q_1, q_2, \dots\}$ except the conditions that R_0 be infinite and $q_i \in R_0$ for every i. Let $n > 0$ and assume inductively that for $i < n$ the rings R_i and elements $p_i \in T$ have been constructed so that the following conditions hold:

(1) $(R_i, R_i \cap M)$ is a subring of T;
(2) R_i is (infinitely) countable for $i > 0$;
(3) p_i is an associate of q_i;
(4) $p_j \in R_i$ for $j \leq i$;

(5) $(Q_j \backslash p_j T)R_i \cap R_i = \{0\}$ for $j \leq i$;
(6) $R_i \cap P = (0)$ and if P' is an associated prime ideal of T, then $R_i \cap P' = (0)$;
(7) $R_i \cap Q_j = (0)$ for $j > i$.

We now work to define the ring R_n and the element p_n. Use Lemma 4 from [7] to find a unit t_n satisfying $q_n t_n + Q$ is transcendental over $R_{n-1}/(Q \cap R_{n-1})$ for all $Q \in \{P\} \cup \{(0)\} \cup \operatorname{Ass} T \cup C \backslash \{Q_n\}$. Let $p_n = q_n t_n$, $S = R_{n-1}[p_n]$ and $R_n = S_{(S \cap M)}$. We claim that R_n and p_n satisfy the above conditions (1)–(7) with i replaced by n. We will show that $(Q_n \backslash p_n T)R_n \cap R_n = \{0\}$ and leave the rest of the conditions to the reader.

We first show $Q_n \cap R_n = p_n R_n$. Let $f \in Q_n \cap R_{n-1}[p_n]$. Then $f = r_k(p_n)^k + \cdots + r_1 p_n + r_0$ where $r_l \in R_{n-1}$. So we have that $f - r_0 \in p_n T \subseteq Q_n$. Hence, $r_0 \in Q_n \cap R_{n-1} = (0)$. It follows that $f \in p_n R_{n-1}[p_n]$. From this, we get that $R_n \cap Q_n = p_n R_n$ as desired.

Now suppose $f \in (Q_n \backslash p_n T)R_n \cap R_n$ and $f \neq 0$. Then $f = qs$, where $q \in Q_n$, $q \notin p_n T$ and $s \in R_n$. We can write $f = p_n^t f'$ and $s = p_n^k s'$, where $f', s' \in R_n$ and $f', s' \notin p_n T$. So we have $p_n^t f' = q p_n^k s'$. Now if $k \geq t$, we have $f' \in Q_n \cap R_n = p_n R_n$, a contradiction. So, $k < t$. It follows that $qs' \in p_n T$. As $p_n T$ is prime and $q \notin p_n T$, we have that $s' \in p_n T$, a contradiction. So we have that $(Q_n \backslash p_n T)R_n \cap R_n = \{0\}$.

Now, letting $S = \bigcup_{i=0}^{\infty} R_i$ if C is infinite, and $S = \bigcup_{i=0}^{k} R_i$ if C contains $k < \infty$ elements, it is not hard to show that S is the desired PC-subring of T. □

Lemma 2.23. *Let (T, M), C, P, and $\{p_i\}$ be as in Definition 2.1. Let R_0 be the prime subring of T and $R_i = R_0[p_1, p_2, \ldots, p_i]$ for $i = 1, 2, \ldots$. Define $S = \bigcup_{i=0}^{\infty} R_i$ if C is infinite, and $S = \bigcup_{i=0}^{k} R_i$ if C contains $k < \infty$ elements. Suppose $S \cap P = (0)$, $S \cap P' = (0)$, whenever P' is an associated prime ideal of T and for each i, $(Q_i \backslash p_i T)S \cap S = \{0\}$. Then there exists a PC-subring of T with respect to the set $\{p_1, p_2, \ldots\}$.*

Proof. If C is infinite, we use Lemma 2.2 to see that $S_{(S \cap M)}$ is a PC-subring of T. On the other hand, suppose C contains $k < \infty$ elements. Then S may be finite. Let $G = \operatorname{Ass} T \cup \{P\} \cup \{(0)\} \cup C$. For each $Q \in G$, let $D_{(Q)}$ be a full set of coset representatives of the cosets $t + Q$ that are algebraic over $R/(R \cap Q)$. Let $D = \bigcup_{Q \in G} Q$, and note that $|D| \leq |T/M|$. Now use Lemma 2.9 with $I = M$ to find an $x \in M$ such that $x \notin \bigcup \{r + Q \mid r \in D, Q \in G\}$. Let $S' = S[x]$ and $S'' = S'_{(S' \cap M)}$. Then S'' is a PC-subring of T with respect to the set $\{p_1, p_2, \ldots, p_k\}$. As the proof of this is similar to other proofs in this article, we leave the details of the proof to the reader. □

Theorem 2.24 is our main result.

Theorem 2.24. *Let (T, M), C, P, and $\{p_i\}$ be as in Definition 2.1 such that the following extra conditions also hold:*

(1) *For each i if $Q \in \operatorname{Ass}(T/p_i T)$, we have $Q \subseteq Q_i$;*
(2) *T is a UFD;*
(3) *$|T| = |T/M|$;*
(4) *T contains the rationals;*
(5) *A PC-subring $(R, R \cap M)$ exists;*
(6) *T_P is a regular local ring and for all i, T_{Q_i} and $(T/p_i T)_{Q_i}$ are regular local rings.*

Then there exists an excellent local UFD $A \subseteq T$ such that:

(1) $p_i \in A$ *for all* i;
(2) $\widehat{A} = T$;
(3) $A \cap P = (0)$, *and if* J *is a prime ideal of* T *with* $J \cap A = (0)$, *then* $J \subseteq P$ *or* $J \subseteq Q_i$ *for some* i;
(4) *For each* i, p_iA *is a prime ideal in* A *and has a local formal fiber with maximal ideal* Q_i.

Proof. The proof is based on the proof of Lemma 3.16 in [3]. First use Theorem 2.20 to construct the ring A. Then all conclusions are clear except that A is excellent. Since T is a domain, A is formally equidimensional. It follows that A is universally catenary. So we must only show that the formal fibers of A are geometrically regular.

Let Q be a nonzero prime ideal of A with $Q \neq p_iA$ for all i. Since $P \cap A = (0)$, we have that $QT \not\subseteq P$. Suppose $QT \subseteq Q_i$ for some i. Then $Q = QT \cap A \subseteq Q_i \cap A = p_iA$, a contradiction. By the construction of A, it follows that the map $A \longrightarrow T/QT$ is onto and so $A/Q \cong T/QT$. Now, let $k(Q) = A_Q/QA_Q$. Then

$$T \otimes_A k(Q) \cong (T/QT)_{\overline{A-Q}} \cong (A/Q)_{\overline{A-Q}} \cong A_Q/QA_Q = k(Q),$$

a field. Also note that if L is a finite field extension of $k(Q)$ then we have that

$$T \otimes_A L \cong T \otimes_A k(Q) \otimes_{k(Q)} L \cong k(Q) \otimes_{k(Q)} L \cong L,$$

also a field. It follows that the fiber over Q is geometrically regular.

We now show that the fiber over the zero ideal of A is geometrically regular. By the way we constructed A, if Q is a prime ideal of T with $Q \cap A = (0)$, then $Q \subseteq P$ or $Q \subseteq Q_i$ for some i. Now $T \otimes_A k((0))$ localized at Q is isomorphic to T_Q. Since T_P and T_{Q_i} are assumed to be regular local rings and $Q \subseteq P$ or $Q \subseteq Q_i$ for some i, we have that T_Q is a regular local ring. Since T contains the rationals, $k((0))$ is a field of characteristic zero. It follows that the fiber over the zero ideal of A is geometrically regular.

It is left to show that the fibers over p_iA are geometrically regular. By the way we constructed A, we have that $T \otimes_A k(p_iA)$ is a local ring with maximal ideal Q_i. Now, $T \otimes_A k(p_iA)$ is isomorphic to $(\frac{T}{p_iT})_{\overline{A-p_iA}}$, and so we have that the ring $T \otimes_A k(p_iA)$ localized at Q_i is isomorphic to $(\frac{T}{p_iT})_{Q_i}$ which is a regular local ring by assumption. Since T contains the rationals, $k(p_iA)$ is a field of characteristic zero, and it follows that the formal fiber of p_iA is geometrically regular. Therefore, A is excellent. $\square$

By the proof of Theorem 2.24, we know exactly what the formal fibers of A are. Specifically, the formal fiber of A at p_iA is the set of prime ideals of T that are contained in Q_i and that contain p_i. The formal fiber of A at (0) is $\{P\}$ union the set of prime ideals of T that are contained in Q_i for some i, but do not contain p_i. Now, suppose that Q is a nonzero prime ideal of A with $Q \neq p_iA$ for every i. Then we have by the above proof that the formal fiber ring of A at Q, namely, $T \otimes_A k(Q)$, is a field. So, there is only one element in the formal fiber of A at Q. In fact, we know that $QT \cap A = Q$ and since $A/Q \cong T/QT$, we have that QT is a prime ideal of T. So the only element in the formal fiber of A at Q is, therefore, QT.

In light of Theorem 2.18, we have the following theorem for the case when C is finite.

Theorem 2.25. *Let (T, M), C, P, and $\{p_i\}$ be as in Definition 2.1 such that the following extra conditions also hold:*

(1) *For each i if $Q \in \mathrm{Ass}(T/p_iT)$, we have $Q \subseteq Q_i$;*
(2) *C is a finite set;*
(3) $|T| = |T/M|$;
(4) *T contains the rationals;*
(5) *A PC-subring $(R, R \cap M)$ exists;*
(6) *T_P is a regular local ring and for all i, T_{Q_i}, and $(T/p_iT)_{Q_i}$ are regular local rings.*

Then there exists an excellent local domain $A \subseteq T$ such that:

(1) *$p_i \in A$ for all i;*
(2) $\widehat{A} = T$;
(3) *$A \cap P = (0)$ and if J is a prime ideal of T with $J \cap A = (0)$, then $J \subseteq P$ or $J \subseteq Q_i$ for some i;*
(4) *For each i, p_iA is a prime ideal in A and has a local formal fiber with maximal ideal Q_i.*

Proof. Proceed as in the proof of Theorem 2.24 using Theorem 2.21 in place of Theorem 2.20. □

We end with an example showing that there is a complete local ring T satisfying the hypotheses of Theorem 2.24.

Example 2.26. Let $T = \mathbb{C}[[x_1, x_2, \ldots, x_n]]$ with $n \geq 3$. Define $P = (x_3, \ldots, x_n)$ and for $i = 1, 2, \ldots$, define $q_i = x_1 - ix_2$ and $Q_i = (x_1 - ix_2, x_3, \ldots, x_n)$. Then by Lemma 2.22, there exists a PC-subring of T with respect to a set $\{p_1, p_2, \ldots\}$, where p_i is an associate of q_i for every i. It follows from Theorem 2.24 that there exists an excellent local UFD A such that $\alpha(A) = \alpha(A/p_iA) = n - 2$ for all i.

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