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Large lower bounds for the betti numbers of graded modules with low regularity

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Abstract
Suppose that M is a finitely-generated graded module (generated in degree 0) of codimension $c \geq 3$ over a polynomial ring and that the regularity of M is at most $2a-2$ where $a \geq 2$ is the minimal degree of a first syzygy of M. Then we show that the sum of the betti numbers of M is at least $\beta_0(M)(2^c + 2^{c-1})$. Additionally, under the same hypothesis on the regularity, we establish the surprising fact that if $c \geq 9$ then the first half of the betti numbers are each at least twice the bound predicted by the Buchsbaum-Eisenbud-Horrocks rank conjecture: for $1 \leq i \leq \frac{c+1}{2}$, $\beta_i(M) \geq 2\beta_0(M)\binom{c}{i}$.

Keywords Betti number · Boij–Söderberg Theory · Buchsbaum-Eisenbud-Horrocks rank conjecture · Total rank conjecture

1 Introduction

Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k and let M be a finitely generated graded S-module of finite length. The total betti number $\beta(M) := \beta_0(M) + \cdots + \beta_n(M)$ is defined to be the sum of the betti numbers of M. This number has been of recent interest, most notably in the context of the Total Rank Conjecture which predicts that $\beta(M) \geq 2^n$. If $\mathrm{char}(k) \neq 2$, this conjecture was recently proved by Walker [8], who also showed that equality holds if and only if M is isomorphic to S modulo a regular sequence—such modules are called complete intersections.

Evidently if M is not a complete intersection, then $\beta(M) > 2^n$ and since $\beta(M)$ must be even, it follows that $\beta(M) \geq 2^n + 2$. In fact, there is reason to believe that if M is not a complete intersection then $\beta(M)$ must be considerably larger than 2^n. It was asked by Charalambous, Evans, and Miller in [3] whether it is true that $\beta(M) \geq 2^n + 2^{n-1}$. They proved that this is the case for arbitrary graded modules M when $n \leq 4$ and for all n when

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M is multi-graded. We remark that if M is not of finite length, then the natural extension is to claim that

$$\beta(M) \geq 2^c + 2^{c-1} \quad \text{where } c \text{ is the codimension of M.} \tag{1.1}$$

Such an extension has recently been obtained for monomial ideals in [2] where it was also proved that equality is possible for all $c \geq 2$. The impetus of this paper was to prove that (1.1) holds for arbitrary M provided that the regularity of M is small relative to the degrees of its first syzygies. This is the content of our first theorem.

Theorem 1.1 *Let M be a graded S-module of codimension $c \geq 3$ generated in degree 0 and let $a \geq 2$ be the minimal degree of a first syzygy of M. If $\operatorname{reg}(M) \leq 2a - 2$, then*

$$\beta(M) \geq \beta_0(M)(2^c + 2^{c-1}).$$

Our result is an extension of work by Erman [5], where he proved, under the same hypothesis on the regularity, that $\beta_i(M) \geq \beta_0(M)\binom{c}{i}$. Erman's work proves a special case of the Buchsbaum-Eisenbud-Horrocks rank conjecture which predicts that $\beta_i(M) \geq \binom{c}{i}$. Naturally, Erman's bound will imply that $\beta(M) \geq \beta_0(M)2^c$ when the regularity hypothesis holds. Noting that $2^c + 2^{c-1} = (1.5)(2^c)$, the stronger bound in Theorem 1.1 asserts that on average, each betti number $\beta_i(M)$ is at least 1.5 times $\beta_0(M)\binom{c}{i}$.

We view the most interesting aspect of our paper to be the discovery uncovered by our proof of Theorem 1.1 which sheds new light on the relative size of betti numbers of modules with low regularity. As a first approach to Theorem 1.1, we considered whether it might be possible to show that each betti number satisfies $\beta_i(M) \geq 1.5\beta_0(M)\binom{c}{i}$. It is easy to see that such a result cannot hold—for instance any Gorenstein algebra S/I will have $\beta_0(S/I) = \beta_c(S/I) = 1$. More generally, it is not hard to find examples where $\beta_i(M) < 1.5\beta_0(M)\binom{c}{i}$ for several values of i. However, quite surprisingly for us, we could find no examples where this occurred for $1 \leq i \leq \lceil \frac{c}{2} \rceil$ (provided that $c \geq 9$). In fact, we are able to prove that the first half of the betti numbers are not only at least 1.5 times Erman's bound, but they are at least double!

Theorem 1.2 *Let M be a graded S-module of codimension $c \geq 9$ generated in degree 0 and let $a \geq 2$ be the minimal degree of a first syzygy of M. If $\operatorname{reg}(M) \leq 2a - 2$ then for each $1 \leq i \leq \lceil c/2 \rceil$, $\beta_i(M) \geq 2\beta_0(M)\binom{c}{i}$.*

We consider Theorem 1.2 to be our main contribution. Indeed, since this says the first half of the betti numbers are at least twice the Horrocks bound, it easily follows that on average the betti numbers are large enough to ensure that their sum is at least $\beta_0(M)(2^c + 2^{c-1})$. Thus the first theorem follows from the second after an analysis of a small number of cases where $3 \leq c \leq 8$; this is done in Sect. 3.

Theorem 1.2 also implies a rather strong connection between the regularity of M and its first few betti numbers. For instance, this means that if $M = S/I$ where I is of codimension at least 9 and is minimally generated by $c \leq \mu < 2c$ quadrics, then the $\operatorname{reg}(M) \geq 3$. In the Artinian (finite-length) case, since the regularity can be interpreted as the maximal socle degree, we can understand this result as making more precise the idea that having a small number of generators will naturally lead to a high socle degree. Our theorem provides bounds on this relationship which are new (even in the Artinian case).

It seems to us rather bizarre that this theorem (like Erman's results) should depend almost completely on the numerics coming from Boij–Söderberg Theory. This mysterious

behavior is also apparent in McCullough's work in [7] concerning the relationship between the regularity of an ideal and the degrees of half of its syzygies. In this vein, our results can be interpreted as saying that the degree of the first syzygy and the number of syzygies in the first half of the resolution can in some cases force the regularity to be large.

We remark that the regularity bound is actually relaxed enough to include many interesting geometric examples. In [5], Erman presents several examples of modules that satisfy $\operatorname{reg}(M) \leq 2a-2$ including smooth curves embedded by linear systems of high degree, toric surfaces, and Artinian rings $M = S/I$ whose socle degree is relatively low.

We comment now on our methods and how they differ from those of Erman. We begin as he did with standard Boij–Söderberg techniques to write an arbitrary betti diagram as a rational combination of normalized *pure* betti diagrams, whose entries $\pi_i(D)$ are each a function of n positive integers $D = (d_1, \ldots, d_n)$. In Sects. 2 and 3 we show that the proofs of our main theorems reduce to finding lower bounds for $\pi_i(D)$; we need to show that $\pi_i(D) \geq 2\binom{n}{i}$ whenever $i \leq \lceil \frac{n}{2} \rceil$. It's then natural to consider the ratio and attempt to show $\pi_i(D)/\binom{n}{i} \geq 2$. Like Erman we reduce these calculations to the study of a function $F(a, b, e, n, i)$ of 5 variables by proving $\pi_i(D)/\binom{n}{i} \geq F(a, b, e, n, i)$. The parameters a, b, e of F depend on the degree sequence in question. It is here that our analysis differs substantially from that of Erman.

Since Erman was concerned with a uniform bound for all betti numbers, his proof (in our notation) shows that $F(a, b, e, n, i) \geq 1$. Our task, on the other hand, is to focus on the first half of the betti numbers and prove that $F(a, b, e, n, i) \geq 2$ for small i. Since this statement is not true for all i (nor is it true if the codimension n is less than 9) our analysis necessarily proceeds in a delicate way. In addition, if $n \leq 8$, since Theorem 1.1 holds whereas Theorem 1.2 does not, independent techniques are developed to address these cases. What ultimately makes the proofs difficult is simply that the function F is complicated, so finding its minimum requires some care. Moreover, there are a host of cases where our general method fails—these arise primarily when the difference between the regularity of M and the generating degree of a first syzygy of M is very small. The reduction via Boij–Söderberg theory necessitates that we consider all of these cases, as otherwise our results would be significantly weaker.

2 Boij–Söderberg basics

In this section we will review the relevant pieces of Boij–Söderberg theory. Rather than state the theory in its fullest generality, we present only the version we need for our results. We begin with an example.

Example 2.1 Let $S = \mathbb{Q}[x, y, z]$ and take I to be an ideal generated by 5 general quadrics. Set $M = S/I$. Similarly, let ϕ be a 3×10 matrix of general quadrics and let $N = \operatorname{Coker} \phi$. Finally, let $M' = S/(x^2, y^2, z^2, xy)$. The betti diagrams of M, N and M' are given below:

$$\begin{array}{c|cccc} \beta(M) & & & & \\ & 1 & -- & -- & -- \\ & -- & 5 & 5 & -- \\ & -- & -- & -- & 1 \end{array}, \quad \begin{array}{c|cccc} \beta(N) & & & & \\ & 3 & -- & -- & -- \\ & -- & 10 & -- & -- \\ & -- & -- & 15 & 8 \end{array}, \quad \begin{array}{c|cccc} \beta(M') & & & & \\ & 1 & -- & -- & -- \\ & -- & 4 & 2 & -- \\ & -- & -- & 3 & 2 \end{array}$$

We point out that the first two diagrams are **pure** in the sense that each column has at most one nonzero entry. The last betti diagram is not pure since the column representing the second syzygy module has two nonzero entries. Further, note that each of the first

two diagrams is a **sub-diagram** of the third diagram, in the sense that the locations of the nonzero entries of the first two fit inside the third diagram. This will be made explicit in what follows.

Finally, we notice the rather astonishing fact that the third betti diagram (thought of as a matrix) can be written as a positive rational linear combination of the first two diagrams:

$$\beta(M') = \frac{2}{5}\beta(M) + \frac{1}{5}\beta(N).$$

The above example is an instance of the following, which is a summary of the main results in Boij–Söderberg Theory.

> "The betti diagram of an (arbitrary) finite-length module can be written as a positive rational linear combination of pure diagrams."

We now set $S = k[x_1, \dots, x_n]$ and work with finitely generated graded S-modules M. Henceforth all of our modules will be assumed to be generated in degree 0; allowing for shifting, this is tantamount to saying that M is generated in a single degree. If M is a finite length module and each syzygy module of M is generated in a single degree then we will say that M **has a pure resolution** (or that M **is pure**). Note that we require that pure modules have finite length. For a pure module M we let $D : (d_0 = 0) < d_1 < \cdots < d_n$ be the sequence whose i-th entry is the degree of the generators of the i-th syzygy module of M. This increasing sequence of integers D is called the **degree sequence of** M . By $\operatorname{reg}(D)$ we will mean the number $d_n - n$, which corresponds to the regularity of the module M.

Remark 2.2 A finite length module M is pure with degree sequence $D : (d_0 = 0) < d_1 < \cdots < d_n$ if and only if for each $i = 0, \dots, n$, the graded betti numbers of M satisfy

$$\beta_{ij}(M) \neq 0 \iff j = d_i.$$

Remarkably, the betti numbers of pure modules are determined up to scalar multiple. Indeed, if a finite length module M is pure with degree sequence D then there is a scalar $\lambda \in \mathbb{Q}$ so that for all i, the following holds:

$$\beta_i(M) = \beta_{i,d_i}(M) = \lambda \pi_i(D) \text{ with } \pi_i(D) = \frac{d_1 \cdots d_n}{\prod_{i \neq j}(d_i - d_j)}. \tag{2.1}$$

This was first proven by Herzog and Kühl [6] and the equalities above are called the Herzog-Kühl equations. Note that since $\pi_0(D) = 1$ we have that $\lambda = \beta_0(M)$. In order to prove Theorems 1.1 and 1.2 we will study the rational functions π_i and establish the following two theorems.

Theorem 2.3 *Suppose that $n \geq 3$ and $D : 0 < d_1 < \ldots < d_n$ is a degree sequence of length $n+1$ with $d_1 \geq 2$ satisfying $\operatorname{reg}(D) \leq 2d_1 - 2$. Then $\sum \pi_i(D) \geq 2^n + 2^{n-1}$.*

Theorem 2.4 *Let D be a degree sequence of length $n+1$ with $d_1 \geq 2$ and $\operatorname{reg}(D) \leq 2d_1 - 2$.*

- If $n \geq 9$, then for each $1 \leq i \leq \lceil n/2 \rceil$,

$$\pi_i(D) \geq 2\binom{n}{i}.$$

- If $n \in \{6, 7, 8\}$, the same conclusion holds unless
  - $d_1 = 2$ and $\operatorname{reg}(D) = 2$ or
  - $d_1 = 3$ and $\operatorname{reg}(D) = 3$.

Remark 2.5 When $n \in \{6, 7, 8\}$ there are only 36 degree sequences satisfying the regularity hypothesis but to which Theorem 2.4 does not apply. The pure diagrams are those that are subdiagrams of one of the following diagrams:

	0	1	2	3					n
0	⋆	–	–	–	–	–	–	–	–
1	–	⋆	⋆	⋆	⋯	⋯	⋆	⋆	–
2	–	–	⋆	⋆	⋯	⋯	⋆	⋆	⋆

,

	0	1	2	3					n
0	⋆	–	–	–	–	–	–	–	–
1	–	–	–	–	–	–	–	–	–
2	–	⋆	⋆	⋆	⋯	⋯	⋆	⋆	–
3	–	–	⋆	⋆	⋯	⋯	⋆	⋆	⋆

.

The content of Theorems 2.3 and 2.4 is purely numerical. Their connection to our main theorems on betti numbers is achieved via the beautiful results of Boij–Söderberg Theory, developed in [1, 4]. This theory shows that the betti diagram of an arbitrary finite length module can be written as a finite rational linear combination of pure diagrams.

Given a module M, its graded betti numbers $\beta_{ij}(M)$ are often arranged into a **betti-diagram**—thought of as a matrix (typically with the convention that $\beta_{i,i+j}(M)$ is in the ith column and the jth row). With this convention the regularity of M is equal to the index of the bottom row in the diagram. If D is a degree sequence of length $n+1$ then we define $B(D)$ to be the betti diagram with entry $\pi_i(D)$ in column i and row $d_i - i$. By the Herzog-Kühl equations (2.1), if M is a pure module with degree sequence D then the betti diagram of M will be a scalar multiple of $B(D)$.

Example 2.6 We associate to the degree sequence $D = \{0, 2, 4, 5\}$ the following diagrams:

⋆	–	–	–
–	⋆	–	–
–	–	⋆	⋆

,

$B(D)$			
	1	–	–
	–	$\frac{10}{3}$	–
	–	–	5 $\frac{8}{3}$

.

We use stars to emphasize that we care about the positions of the nonzero entries in the diagram, then use $B(D)$ to denote the diagram of numbers $\pi_i(D)$.

Given two diagrams B and B' we say that B' is a **sub-diagram** of B if for each nonzero entry of B', the corresponding entry in B is also nonzero. If B is the betti diagram of a finitely generated module then there are a finite number of degree sequences D such that $B(D)$ is a subdiagram of B. We now summarize the results of Eisenbud-Schreyer and (respectively) Boij–Söderberg [1, 4] which show that a finite length module (respectively, one of codimension c) can be decomposed as a sum of pure diagrams.

Theorem 2.7 (Main Theorem of Boij–Söderberg Theory [1, 4]). *Let M be a finitely generated S-module with betti diagram B. Suppose that* $\operatorname{codim} M = c$. *If $\Omega = \{B(D)\}$ is the set of all pure sub-diagrams of B having between $c+1$ and $n+1$ columns (indexed by their*

degree sequences D with lengths between $c+1$ and $n+1$) then there exist non-negative rational numbers λ_D such that

$$B = \sum_{B(D)\in\Omega} \lambda_D B(D).$$

In particular, this implies that $\beta_0(M) = \sum \lambda_D$ and more generally, $\beta_i(M) = \sum \lambda_D \pi_i(D)$.

3 Reduction to Theorem 2.4

In this section we explain how to deduce our main theorems from their numerical versions stated in Sect. 2. We will then assume Theorem 2.4 and use it to prove Theorem 2.3. For convenience, all four theorems are restated in the diagram below.

Main Theorems on Betti Numbers	Main Numerical Results
Theorem 1.1. Let M be a graded S-module of codimension $c \geq 3$ generated in degree 0 and let $a \geq 2$ be the minimal degree of a first syzygy of M. If $\operatorname{reg}(M) \leq 2a-2$ then $\beta(M) \geq \beta_0(M)(2^c + 2^{c-1})$.	**Theorem** 2.3. Suppose that $n \geq 3$, and D is a degree sequence of length $n+1$, and $d_1 \geq 2$ satisfying $\operatorname{reg}(D) \leq 2d_1 - 2$. Then $\sum \pi_i(D) \geq 2^n + 2^{n-1}$.
Theorem 1.2. Let M be a graded S-module of codimension $c \geq 9$ generated in degree 0 and let $a \geq 2$ be the minimal degree of a first syzygy of M. If $\operatorname{reg}(M) \leq 2a-2$ then for each $1 \leq i \leq \lceil c/2 \rceil$, $\beta_i(M) \geq 2\beta_0(M)\binom{c}{i}$.	**Theorem** 2.4. If $d_1 \geq 2$ and $\operatorname{reg}(D) \leq 2d_1 - 2$ and $n \geq 9$ then for each $1 \leq i \leq \lceil n/2 \rceil$, $\pi_i(D) \geq 2\binom{n}{i}$. If $n \in \{6,7,8\}$ and either $d_1 \geq 3$ or $\operatorname{reg}(D) - d_1 + 1 \neq 1$, then the same conclusion holds.

The theorems on the left follow more or less immediately from the corresponding theorems on the right via Boij–Söderberg theory. With the exception of a small number of special cases when $n < 9$, Theorem 2.3 will follow from Theorem 2.4, the proof of which will be postponed until Sect. 4.

3.1 Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1 Suppose M is generated in degree zero, and $a \geq 2$ is the minimal degree of a first syzygy of M. By Theorem 2.7 there exist nonnegative rational numbers a_D such that

$$\beta_i(M) = \sum_D a_D \pi_i(D) \tag{3.1}$$

where D runs over all degree sequences of length $\ell(D) \in [c+1, n+1]$ whose betti diagrams, $B(D)$, are sub-diagrams of $B(M)$. Let D be such a degree sequence. Then $d_1 \geq a$ and as we have assumed $\operatorname{reg} M \leq 2a-2$, it follows that

$$\operatorname{reg}(D) = d_{\ell(D)} - \ell(D) \leq \operatorname{reg} M \leq 2a - 2 \leq 2d_1 - 2.$$

Hence we can apply Theorem 2.3. Since every degree sequence appearing in the sum has length at least $c+1$, Theorem 2.3 implies that $\sum_i \pi_i(D) \geq 2^c + 2^{c-1}$. Hence we have

$$\beta(M) = \sum_{i=0}^{n} \beta_i(M) = \sum_D a_D \left(\sum_{i=0}^{n} \pi_i(D) \right) \geq \sum_D a_D(2^c + 2^{c-1}) = \beta_0(M)(2^c + 2^{c-1}).$$

□

Proof of Theorem 1.2 The scaffolding is exactly the same as in the previous proof. If $c \geq 9$ then Eq. (3.1) and Theorem 2.4 imply for $i \in \{1, \ldots, \lceil c/2 \rceil\}$

$$\beta_i(M) = \sum_D a_D \pi_i(D) \geq \sum_D a_D \, 2\binom{c}{i} = \beta_0(M)\binom{c}{i}.$$

□

3.2 Proof of Theorem 2.3

Proof of Theorem 2.3 when Theorem 2.4 holds Suppose that D is a degree sequence satisfying the hypotheses of Theorem 2.4. We recall that Erman proved that $\pi_i \geq \binom{n}{i}$ for all i [5, Theorem 1.2]. Now let us add up all of the π_i in pairs: $\pi_i + \pi_{n-i}$. If n is odd, there are an even number of pairs. Now $\pi_0 + \pi_n \geq 2$ since $\pi_0 = 1$ and $\pi_n \geq 1$ by Erman's result. In all other pairs, we combine Theorem 2.4 with Erman's result, and conclude that $\pi_i + \pi_{n-i} \geq 3\binom{n}{i}$. Moreover, since the assumption on indices in Theorem 2.4 includes $i = \lceil n/2 \rceil$, the last pair is at least $4\binom{n}{(n-1)/2}$. Thus

$$\sum \pi_i \geq 2 + 3\binom{n}{1} + \cdots + 3\binom{n}{\frac{n-1}{2}} + \binom{n}{\frac{n-1}{2}} \geq 2 + \frac{3}{2}(2^n - 2) + \binom{n}{\frac{n-1}{2}} \geq 2^n + 2^{n-1}.$$

When n is even, we proceed by pairing terms exactly as before. In this case however, there is a central term in the sum (the term $\pi_{n/2}$) which has no companion. We thus have:

$$\begin{aligned} \sum \pi_i \geq & 2 + 3\binom{n}{1} + \cdots 3\binom{n}{\frac{n-2}{2}} + 2\binom{n}{\frac{n}{2}} \geq 2 + \frac{3}{2}\left(2^n - 2 - \binom{n}{\frac{n}{2}}\right) + 2\binom{n}{\frac{n}{2}} \\ & \geq 2^n + 2^{n-1}. \end{aligned}$$

□

Proof of Theorem 2.3 for $n \in \{6, 7, 8\}$

By Remark 2.5 there are only 36 degree sequences D that satisfy $d_1 \geq 2$ and $\operatorname{reg}(D) \leq 2d_1 - 2$ for which Theorem 2.4 does not apply. Using Macaulay2 we checked that the sum of $\pi_i(D)$ in each of these cases is at least $2^n + 2^{n-1}$. The reader is directed to the file computations.m2 included in our arXiv posting for explicit code that can be used to verify this statement. □

Proof of Theorem 2.3 for $n \in \{3, 4, 5\}$

For each value of n, we will verify that $\sum \pi_i \geq 1.5 \cdot 2^n$ via a direct computation. Suppose first that $n = 3$ so that the degree sequence $D = \{0, d_1, d_2, d_3\}$. We change notation to emphasize the nonlinear parts of D by instead writing it as

$D = \{0, a, a+x+1, a+x+y+2\}$, where $x, y \geq 0$ can easily be computed from the d_i's. The Herzog-Kühl equations (2.1) are explicit formulae for the $\pi_i(D)$'s in terms of the d_i's:

$$\pi_0(D) = 1, \qquad \pi_1(D) = \frac{(a+x+1)(a+x+y+2)}{(x+1)(x+y+2)},$$
$$\pi_2(D) = \frac{a(a+x+y+2)}{(x+1)(y+1)}, \quad \pi_3(D) = \frac{a(a+x+1)}{(x+y+2)(y+1)}.$$

We may assume $a \geq 2$ and our regularity assumption says $x+y+1 \leq a$. We want to prove that $\sum_{i=0}^{3} \pi_i(D) \geq 12$, which after clearing denominators is equivalent to:

$$a^2 + ax + ay + 2a - 5xy - 5x - 5y - 5 \geq 0.$$

If $x = y = 0$ so that the resolution is linear, then the assumption that $a \geq 2$ implies the inequality holds. On the other hand if the resolution is not linear, the left hand side is clearly an increasing function of a, so it suffices to consider the case that $a = x+y+1$, whereby the inequality becomes

$$0 \leq 2x^2 + 2y^2 - 2xy - 2 = (x-y)^2 + x^2 + y^2 + xy - 2.$$

Evidently, each of these terms is positive at least two are nonzero (since x and y are not both 0), so the inequality holds as desired.

Repeating an identical analysis with $n = 4$ (so that $D = \{0, a, a+x+1, a+x+y+2, a+x+y+z+3\}$) again results in a polynomial inequality for which the left hand side is an increasing function of a. After considering the linear case separately, we set $a = x+y+z+1$, and are left to verify the polynomial inequality

$$\begin{aligned} &2x^4 + 5x^3y + 4x^2y^2 + xy^3 + 7x^3z + 9x^2yz + 4xy^2z + 2y^3z + 9x^2z^2 + 8xyz^2 \\ &\quad + 5y^2z^2 + 5xz^3 + 4yz^3 + z^4 + 12x^3 + 19x^2y + 10xy^2 + 3y^3 + 27x^2z + 15xyz \\ &\quad + 12y^2z + 23xz^2 + 17yz^2 + 8z^3 + 22x^2 + 13xy + 9y^2 + 23xz + 12yz + 17z^2 + 6x \\ &\quad + 4z - 6 \geq 0 \end{aligned}$$

This will hold provided not all of $x, y, z = 0$.

The proof strategy for $n = 5$ is exactly the same and begins by setting $D = \{0, a, a+x+1, a+x+y+2, a+x+y+z+3, a+x+y+z+w+4\}$, then using the Herzog-Kühl equations to get a polynomial inequality. The expression thus obtained is now too complicated to be analyzed by hand, though it's still very manageable for a machine. By writing it as a polynomial in a, one can verify that all of the coefficients (besides the constant term) are positive and therefore that left hand side is increasing as a function of a. Again substituting $a = x+y+z+w+1$, one obtains an expression and factors it (with a computer) to arrive at an inequality in which all terms on the left hand side are positive except for the constant term. A simple computer verification shows that the inequality holds for all $x, y, z, w \geq 0$. □

Remark 3.1 The file computations.m2 included in our arXiv posting contains code to verify the numerical statements in this paper. All statements that need computer verification occur in consideration of low codimension cases and have now been made.

4 Proof of Theorem 2.4

In this section we will prove Theorem 2.4, which is the last ingredient needed to complete the proofs of our main results. We endeavor to show that for suitable D and i, we have

$$\pi_i(D) \geq 2\tbinom{n}{i}.$$

Thus it is natural to study the function $(D, i) \mapsto \pi_i(D)/\binom{n}{i}$. Of course this function depends on $n+1$ parameters, so a simplification is required before a reasonable analysis can be performed. We will define a function F depending on five parameters such that

$$\frac{\pi_i(D)}{\binom{n}{i}} \geq F(a, b, e, n, i).$$

Main Notation: Let $D : 0 < d_1 < \cdots < d_n$ and set $a = d_1$. Given $i \geq 1$, we define a modification of D as follows:

$$D^i = \{0, a, a+1, a+2, \ldots, a+(i-2), d_i, d_n-(n-i-1), d_n-(n-i-2), \ldots d_n\}. \tag{4.1}$$

We now consider a degree sequence D^i, as in Eq. (4.1) above, and focus our attention on its nonlinear parts:

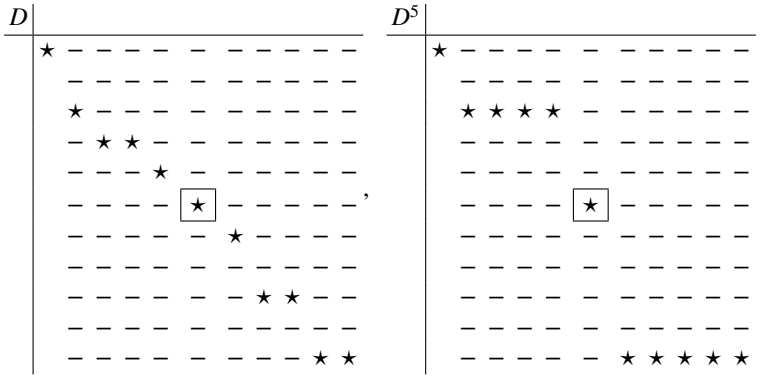
$$b := d_i - a - i + 1, \quad e := d_n - d_i - n + i.$$

Notice then that we have

$$d_n = a + b + e + n - 1, \qquad \text{and} \qquad \operatorname{reg}(D) = a + b + e - 1.$$

The reader may want to ignore these equations and press on to the example that follows, which should clarify the idea (and resolve the ambiguity when $i = 1$).

Example 4.1 Suppose that $i = 5$ and $D = \{0, 3, 5, 6, 8, \boxed{10}, 12, 15, 16, 19, 20\}$ then the betti diagrams for D and D^5 would be formatted as shown



Visually, we have kept d_i in the same place, but have shifted all of the earlier numbers to the top of the diagram and all of the later ones to the bottom. Notice that in this example $a = 3$. In the right-hand diagram there are visible jumps of size $b = 3$ and $e = 5$ on either side of the $\boxed{\star}$ in position i.

Lemma 4.2 *If $D : 0 < (d_1 = a) < d_2 < \cdots < d_n$ is a degree sequence then for all $i \geq 1$*

$$\pi_i(D) \geq \pi_i(D^i).$$

Proof We prove a slightly more general statement. Let $i \geq 1$ and suppose that $D' = \{0, d'_1, \ldots, d'_n\}$ is a degree sequence with $d'_i = d_i$. Then

$$\pi_i(D) = \prod_{j \neq i} \frac{d_j}{|d_j - d_i|}, \quad \pi_i(D') = \prod_{j \neq i} \frac{d'_j}{|d'_j - d_i|}.$$

As all the terms in the product are positive, a sufficient condition for $\pi_i(D) \geq \pi_i(D')$ is that

$$\frac{d_j}{|d_j - d_i|} \geq \frac{d'_j}{|d'_j - d_i|}$$

for all $j \neq i$. If $j < i$ then this is equivalent to requiring

$$\frac{d_j}{d_i - d_j} \geq \frac{d'_j}{d_i - d'_j} \quad \iff \quad d_j \geq d'_j.$$

Conversely, if $j > i$ then the inequality is $d_j \leq d'_j$. To conclude, we simply observe that all of these inequalities hold for $D' = D^i$, whence the result follows. □

We now compute

$$\begin{aligned}
\pi_i(D^i) &= \frac{a(a+1)\cdots(a+(i-2))}{(b+1)(b+2)\cdots(b+(i-1))} \frac{(a+b+e+i)\cdots(a+b+e+n-1)}{(e+1)(e+2)\cdots(e+n-i)} \frac{i!(n-i)!}{n!} \binom{n}{i} \\
&= \frac{a(a+1)\cdots(a+(i-2))}{(b+1)(b+2)\cdots(b+(i-1))} \frac{(n+a+b+e-1)!e!}{(a+b+e+i-1)!(n-i+e)!} \frac{i!(n-i)!}{n!} \binom{n}{i} \\
&= \frac{(a)\cdots(a+(i-2))}{(b+1)\cdots(b+(i-1))} \frac{(n+1)\cdots(n+a+b+e-1)}{(i+1)\cdots(i+a+b+e-1)} \frac{e!}{(n-i+1)\cdots(n-i+e)} \binom{n}{i}.
\end{aligned}$$

Definition 4.3 We define the function $F = F(a, b, e, n, i)$ as the coefficient of $\binom{n}{i}$ in the above computation. The domain of F is $b \geq 0$, $e \geq 0$, $a \geq 2$, $n \geq 3$, $1 \leq i \leq n$,

$$F(a,b,e,n,i) = \frac{(a)\cdots(a+(i-2))}{(b+1)\cdots(b+(i-1))} \frac{(n+1)\cdots(n+a+b+e-1)}{(i+1)\cdots(i+a+b+e-1)} \frac{e!}{(n-i+1)\cdots((n-i)+e)}.$$

In the sequel we will refer to each of the three fractions in the above equation as a grouping. When $i = 1$ there are no terms in the first grouping. Similarly, when $e = 0$ the third grouping is empty.

Our present goal is to show that $F(a, b, e, n, i)$ is at least 2 for a suitable range of inputs (e.g. $i \leq \lceil n/2 \rceil$). At this point we direct the reader to this video https://tinyurl.com/LargeLowerBounds, which explains how our analysis of the function F will proceed below. While the remainder of the paper is technical, the underlying idea is rather simple and is easily conveyed

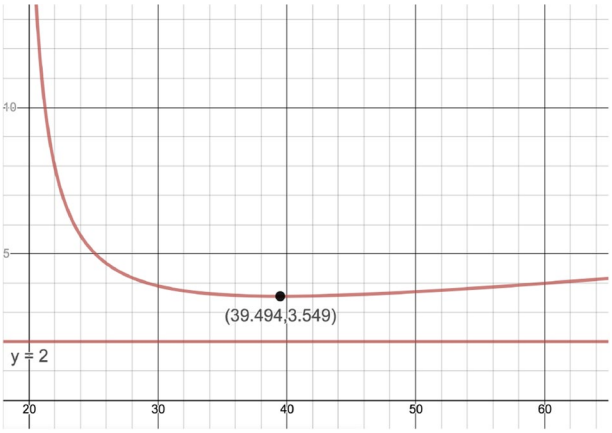


Fig. 1 A graph of $F(2, 0, 1, n, 20)$ as a function of n

in the linked video. The proof of Theorem 2.4 includes a flowchart (Fig. 2) that explains how the lemmas and computations that follow fit together.

Lemma 4.4 *F is increasing as a function of a*:

$$F(a, b, e, n, i) \leq F(a + 1, b, e, n, i).$$

Proof If $i = 1$, then $F(a + 1, b, e, n, i)$ is equal to $F(a, b, e, n, i)$ times an additional factor which has the form $(s + n + a)/(s + i + a)$ for some $s \in \mathbb{N}$, which is evidently at least 1. If $i > 1$, then in addition to this extra factor, the numerators of the terms in the first grouping in $F(a + 1, b, e, n, i)$ will be larger than the corresponding terms of $F(a, b, e, n, i)$. □

One might hope that F is an increasing function of n. This is not the case as evidenced by Fig. 1 which shows the graph of $y = F(2, 0, 1, n, 20)$. Notice however, that for large enough n, the function F is increasing. This is no coincidence, as the following lemma shows.

Lemma 4.5 *If $n \geq 2i - 1$ and $\operatorname{reg}(D) \leq 2a - 2$ then*

$$F(a, b, e, n, i) \leq F(a, b, e, n + 1, i).$$

That is, if i is at most $\frac{n+1}{2}$ then F is an increasing function of n.

Proof Let $R = \operatorname{reg}(D) = a + b + e - 1$. Using our assumption on the regularity, we have

$$2a - 2 \geq a + b + e - 1 \quad \Longrightarrow \quad a - 1 \geq b + e.$$

This in turn implies

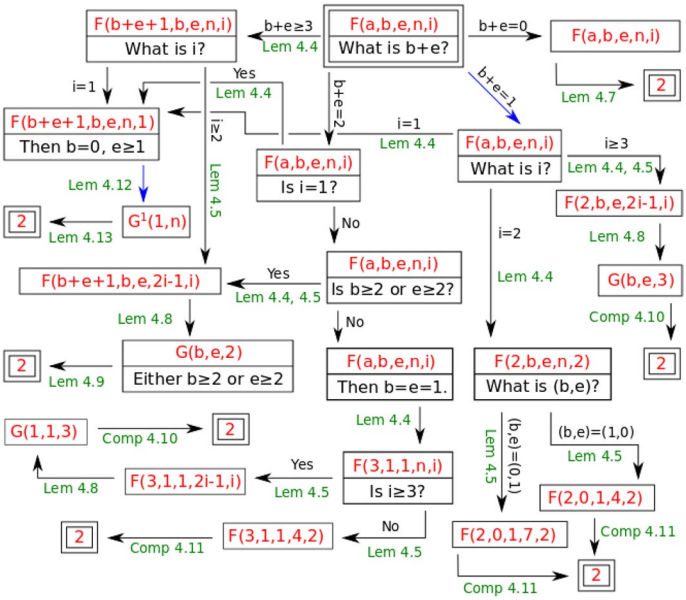


Fig. 2 The proof when $n \geq 9$. The red expression in each box is the current lower bound for $F(a, b, e, n, i)$; the black question tells one how to proceed. Arrows are decorated with the possible answers to the questions (in black) and the lemma or computation used (in green) to obtain the new lower bound (i.e. arrows can be read as $\geq$ symbols). The two blue arrows highlight the places where our argument differs for $n \in \{6, 7, 8\}$

$$R = b + e + a - 1 \geq 2(b + e) \geq 2e.$$

Further if $n \geq 2i - 1$ then

$$\begin{aligned} n \geq & (2i-1)\frac{R-e}{R-e} = \frac{2iR - R - 2ie + e}{R-e} = \frac{(i-1)R + e + i(R-2e)}{R-e} \\ \geq & \frac{(i-1)R + e}{R-e}. \end{aligned}$$

Finally we compute

$$\frac{F(a,b,e,n+1,i)}{F(a,b,e,n,i)} = \frac{(n+a+b+e)(n-i+1)}{(n+1)((n-i)+e+1)}.$$

This will be at least 1 provided

$$(n+a+b+e)(n-i+1) \geq (n+1)((n-i)+e+1)$$

which is equivalent to:

$$n \geq \frac{(i-1)R+e}{R-e}.$$

This is the inequality we have shown above. □

Remark 4.6 Notice that Fig. 1 shows that we cannot improve the bound $n \geq 2i-1$. Further, note that in this proof we used that $\operatorname{reg}(D) \geq 2e$ and that this came from our assumption that $\operatorname{reg}(D) \leq 2a-2$. If we relax that bound, even by one, say to $2a-1$ then it will not be true that F is an increasing function of n. For instance, consider the following two degree sequences (with $a=2, b=0, e=2, i=3, R=3$):

$$\{0,2,3,4,7,8\}\ ,\ \{0,2,3,4,7,8,9\}$$
$$F(a,b,e,5,i) > \ F(a,b,e,6,i).$$

We have just seen (Lemmas 4.4 and 4.5) two crucial observations about the function F. Using these, a few elementary computations allows us to establish Theorem 2.4 for the vast majority of degree sequences of pure diagrams. However, to obtain the full strength of Theorem 1.1, our reduction via Boij–Söderberg theory necessitates that we consider *all* degree sequences of pure sub-diagrams of the betti diagram of M and many of these degree sequences are *not* covered by the lemmas above. Thus the analysis that follows requires a careful analysis of the function F.

We begin by addressing the case of linear resolutions. These correspond to the case when $b=e=0$ and are handled by the following lemma.

Lemma 4.7 *If $b=e=0$, then $F(a,0,0,n,i) \geq 2$.*

Proof If $i \geq 2$, then

$$F(a,0,0,n,i) = \frac{a}{1}\frac{(a+1)\cdots(a+(i-2))}{(2)\cdots(i-1)}\frac{(n+1)\cdots(n+a-1)}{(i+1)\cdots(i+a-1)} \geq \frac{a}{1} \geq 2.$$

On the other hand, if $i=1$ there are no terms in the first grouping. Since $a \geq 2$ and $n \geq 3$, there is at least one term in the middle grouping and we have

$$F(a,0,0,n,1) = \frac{(n+1)(n+2)\cdots(n+a-1)}{(2)(3)\cdots(a)} \geq \frac{(n+1)}{(2)} \geq \frac{4}{2} = 2.$$

□

Our approach is now as follows: by Lemma 4.7 we may assume that $b+e \geq 1$. For fixed b, e, n, i our regularity assumption provides a minimum possible value of a: we have $a+b+e-1 \leq 2a-2$ and thus $a \geq b+e+1$. In light of Lemma 4.4, it's natural to set $a=b+e+1$. We can then apply Lemma 4.5 and decrease n to its minimum possible value of $n=2i-1$. However we will only do this when $i \geq 2$, since we only want to consider degree sequences with $n \geq 3$; our argument will need modifications when $i=1$. Thus, for $i \geq 2$ and $b+e \geq 1$, we now consider the function $G(b,e,i)$ defined by making these substitutions.

$$G(b,e,i) := F(b+e+1,b,e,2i-1,i)$$
$$= \frac{(b+e+1)\cdots(b+e+(i-1))}{(b+1)(b+2)\cdots(b+(i-1))}\frac{(2i)\cdots(2i+2b+2e-1)}{(i+1)\cdots(i+2b+2e)}\frac{e!}{(i)\cdots(i+e-1)}$$

We remind the reader that our goal is to find a lower bound for $\pi_i(D)$ and point out that at this point we have (for $b+e \geq 1$ and $i \geq 2$):

$$\pi_i(D) \geq \pi_i(D^j) \geq F(a,b,e,n,i) \geq G(b,e,i).$$

Lemma 4.8 *G is an increasing function of i: $G(b,e,i) \leq G(b,e,i+1)$.*

Proof We consider the quotient

$$\frac{G(b,e,i+1)}{G(b,e,i)} = \frac{b+e+i}{b+i}\frac{(2i+2b+2e+1)(2i+2b+2e)(i+1)}{(2i)(2i+1)(i+2b+2e+1)}\frac{i}{i+e}.$$

We want this to be at least 1. When we cross-multiply and subtract we are left with the inequality:

$$4b^3i^2 + 4b^2ei^2 + 4be^2i^2 + 4e^3i^2 + 4b^2i^3 + 4bei^3 + 4e^2i^3 + 4b^3i + 8b^2ei + 8be^2i + 4e^3i$$
$$+ 10b^2i^2 + 14bei^2 + 10e^2i^2 + 6bi^3 + 6ei^3 + 2b^2i + 2bei + 2e^2i + 2bi^2 + 2ei^2 \geq 0$$

which is evident. □

In consideration of this, since $G(b,e,i) \geq G(b,e,2)$ for all $i \geq 2$ we show, with a few minor exceptions, that $G(b,e,2) \geq 2$ for relevant inputs.

Lemma 4.9 *If either $b \geq 2$ or $e \geq 2$, then $G(b,e,2) \geq 2$.*

Proof We simply compute

$$G(b,e,2) = \frac{b+e+1}{b+1}\frac{(4)\cdots(2b+2e+3)}{(3)\cdots(2b+2e+2)}\frac{e!}{(2)\cdots(e+1)}$$
$$= \frac{b+e+1}{b+1}\frac{2b+2e+3}{3}\frac{1}{e+1}.$$

This will be at least 2 if and only if

$$2b^2 - 2be + 2e^2 - b - e - 3 \geq 0.$$

Now

$$2b^2 - 2be + 2e^2 - b - e - 3 = (b-e)^2 + b^2 + e^2 - b - e - 3.$$

If $b = e$ then this is $2b^2 - 2b - 3$ which will be nonnegative provided $b \geq 2$. Otherwise, if either b or e is at least 2 then one of $b^2 - b$ or $e^2 - e$ will be at least 2. Thus if $b \neq e$ then

$$(b-e)^2 - 3 + (b^2 - b) + (e^2 - e) \geq 1 - 3 + 2 = 0.$$

□

Restricting our attention to the situation where $i \geq 2$, the lemmas we have established are sufficient to conclude that $F \geq 2$ for the vast majority of relevant inputs. The remaining cases (still assuming that $i \geq 2$) are treated via direct computation, easily done by hand.

$$G(1,0,3) = 2.1 \qquad G(0,1,3) = 2.1 \qquad G(1,1,3) = 2.4$$

Computation 4.10 As $G(b, e, i)$ is an increasing function of i, these computations will allow us to obtain the desired lower bound on F when $i \geq 3$. Indeed, either Lemma 4.9 applies or else $b + e = 1$ and $G(b, e, i) \geq G(b, e, 3)$ which must be one of the numbers above.

We close with one final computation as well as a discussion of what happens for $i = 1$. The reader may note that the values of n in these computations are creeping upwards; this is the first indication for the source of our hypothesis that n be greater than 9.

$$F(3,1,1,4,2) = 2.33 \qquad F(2,1,0,4,2) = 2.5 \qquad F(2,0,1,7,2) = 2$$

Computation 4.11 We now close by handling the case $i = 1$. Note that $i = 1$ implies that $b = 0$. We may assume that $e > 0$ and the assumption that $\operatorname{reg}(D) \geq 2a - 2$ implies that we may assume $a = b + e + 1 = e + 1$. What remains is to determine when

$$G^1(e,n) := F(e+1,0,e,n,1) \geq 2.$$

Using (4.3), we see that

$$G^1(e,n) = \frac{(n+1)\cdots(n+2e)}{(2)\cdots(1+2e)} \frac{e!}{(n)\cdots(n+e-1)}.$$

There is a finite set of inputs for which $G^1(e,n) < 2$, and these are the source of the 36 betti diagrams of pure modules which satisfy our regularity bound but to which Theorem 2.4 does not apply.

Lemma 4.12 *For all $n \geq 3$ and $e \geq 1$, we have*

$$F(e+1,0,e,n,1) \leq F(e+2,0,e+1,n,1).$$

That is, for all n, the function $G^1(e,n) := F(e+1,0,e,n,1)$ is increasing as a function of e.

Proof As usual, we want to establish the following inequality.

$$\frac{F(e+2,0,e+1,n,1)}{F(e+1,0,e,n,1)} = \frac{(2e+n+2)(2e+n+1)}{(2e+3)(2e+2)} \cdot \frac{(e+1)}{(e+n)} \geq 1.$$

Cross-multiplying, simplifying, and factoring, we find that this equivalent to

$$(n-1)(n-2)(e+1) \geq 0,$$

which is evident as $n \geq 3$ and $e \geq 1$. □

Lemma 4.13 *If $n \geq 9$, then $G^1(1, n) \geq 2$.*

Proof We compute

$$G^1(1, n) = \frac{(n+1)(n+2)}{6} \cdot \frac{1}{n}.$$

This is greater than or equal to 2 if and only if $n^2 - 9n + 2 \geq 0$, which is the case for n at least 9. □

As before, some sporadic cases will be handled by a few direct computations.

$$F(3, 1, 1, 6, 2) = 4.2 \qquad G^1(2, 6) = 2.$$

Computation 4.14 We have need of one final computation that will reduce from infinite to finite the number of degree sequences of pure diagrams that do not satisfy the hypotheses of our theorem. Indeed, if the regularity bound is strengthened by one and we assume that $\operatorname{reg}(D) \leq 2a - \mathbf{3}$, then the minimum possible value of a is $b + e + 2$. We compute:

Computation 4.15 For $i \in \{1, 2, 3, 4\}$ and $(b, e) \in \{(1, 0), (0, 1)\}$, we have $F(3, b, e, 6, i) \geq 2$.

We are now ready to put the jigsaw puzzle together and prove Theorem 2.4. For the reader's convenience, we have restated it below in an equivalent form.

Proposition 4.16 *Let D be a degree sequence with $\operatorname{reg}(D) \leq 2a - 2$ and $n \geq 9$. Then for each $1 \leq i \leq \lceil n/2 \rceil$, $\pi_i(D) \geq 2\binom{n}{i}$. If $n \in \{6, 7, 8\}$ and either $a \neq 2$ or $b + e \neq 1$, then the same conclusion holds.*

Proof of Proposition 4.16 The proof amounts to piecing together the lemmas and computations above and is depicted in the flowchart (Fig. 2). A key point is that for a fixed degree sequence D, while D^i (and the associated nonlinear parts b and e) depends on the value of i, the sum $b + e$ of D^i is a function only of the original degree sequence D and not of i. For $n \geq 9$, refer to the flow chart.

If the resolution is linear so that $b + e = 0$, then Lemma 4.7 applies to give the desired conclusion. If $b + e \geq 3$, then we apply Lemma 4.4 and decrease a to its minimum possible value while maintaining our regularity assumption. Then, if $i \geq 2$, we apply Lemma 4.5, decreasing n to get

$$F(a, b, e, n, i) \geq F(b + e + 1, b, e, n, i) \geq F(b + e + 1, b, e, 2i - 1, i) = G(b, e, i).$$

Since $b + e \geq 3$, either $b \geq 2$ or $e \geq 2$ regardless of the value of i. Thus, in all cases we may apply Lemma 4.8 decreasing the value of i and then apply Lemma 4.9 to conclude

$$F(a, b, e, n, i) \geq G(b, e, i) \geq G(b, e, 2) \geq 2.$$

If $i = 1$, we still apply Lemma 4.4. Then we note that this implies $b = 0$. Now Lemmas 4.12 and 4.13 allows us to conclude

$$F(a, b, e, n, i) \geq F(b + e + 1, b, e, n, i) = F(e + 1, 0, e, n, 1) = G^1(e, n) \geq G^1(1, n) \geq 2.$$

Now if $b + e = 2$, the above argument fails only for those values of i where $b = e = 1$ (because Lemma 4.9 fails); when $i = 1$, the argument needs no modification. If $b = e = 1$ and $i \geq 3$, then we apply Lemmas 4.4, 4.5, and 4.8 just as above only this time we use Computation 4.10 to conlcude

$$F(a, 1, 1, n, i) \geq F(3, 1, 1, n, i) \geq F(3, 1, 1, 2i - 1, i) = G(1, 1, i) \geq G(1, 1, 3) > 2. \tag{4.2}$$

If $i = 2$, then rather than decreasing n to $2i - 1 = 3$ in applying Lemma 4.5, we set $n = 4$ and use Computation 4.11.

$$F(a, 1, 1, n, 2) \geq F(3, 1, 1, n, 2) \geq F(3, 1, 1, 4, 2) \geq 2.$$

If $b + e = 1$, the chain of inequalities (4.2) still holds for $i \geq 3$ and the logic from above still applies for $i = 1$. Thus, the only remaining case is $i = 2$ and our assumptions imply $(b, e) \in \{(0, 1), (1, 0)\}$. When $(b, e) = (0, 1)$ (resp. $(b, e) = (1, 0)$), apply Lemma 4.5 to decrease n to 7 (resp. 4), then apply Computation 4.11 to get

$$F(a, b, e, n, 2) \geq F(2, b, e, n, 2) \geq 2.$$

If $n \in \{6, 7, 8\}$, the proof differs only in a few places and these are depicted in the flow chart by two blue arrows. The arrow on the left hand side concerns the setting where $b + e \geq 3$ and $i = 1$, which implies that $b = 0$ and $e \geq 2$. This time we apply Lemma 4.5 and decrease n to the value of 6, then apply Lemma 4.12 setting $e = 2$ and use Computation 4.14

$$F(a, b, e, n, i) \geq F(e + 1, 0, e, n, 1) \geq F(e + 1, 0, e, 6, 1) = G^1(e, 6) \geq G^1(2, 6) = 2.$$

The second blue arrow concerns the case that $b + e = 1$, and for finitely many degree sequences, our method fails here. If $\operatorname{reg}(D) \leq 2a - 3$, then we apply Lemma 4.4 decreasing a to the minimum possible value of $a = b + e + 1 = 3$. Next apply Lemma 4.5 and set $n = 6$. Noting that $(b, e) \in \{(1, 0), (0, 1)\}$, we use computation 4.15 to obtain

$$F(a, b, e, n, i) \geq F(3, b, e, n, i) \geq F(3, b, e, 6, i) \geq 2.$$

□

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