Blowups of Algebraic Varieties

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July 31, 2008

Introduction

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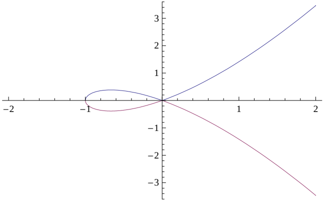

Figure: The Graph of $y^2 = x^2 + x^3$

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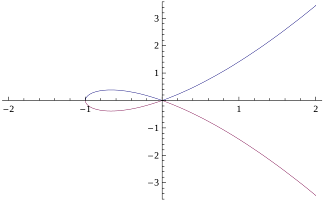

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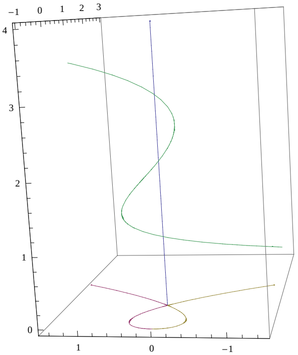

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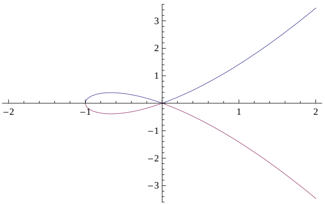

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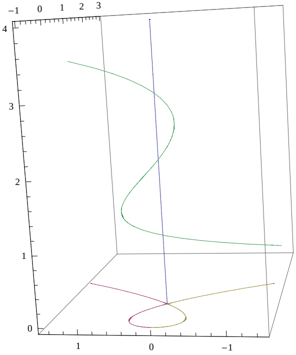

Figure: The Blowup of $y^2 = x^2 + x^3$ at the Origin

Varieties

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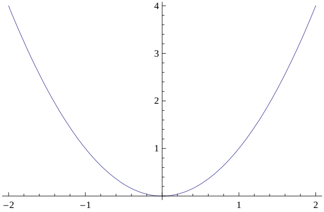

Figure: $\mathbb{V}(y - x^2)$

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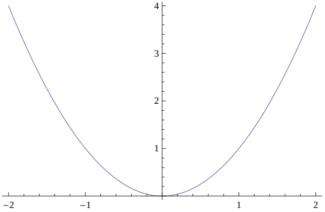

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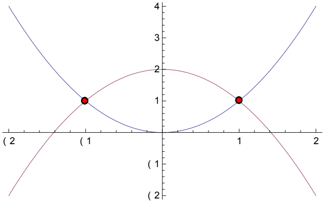

Figure: $\mathbb{V}(y - x^2, y + x^2 - 2) = \{(-1, 1), (1, 1)\}$

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$$f(x,y) = y^2 - x^2 - x^3 : \quad \frac{\partial f}{\partial x} = -2x - 3x^2$$
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Fact

Let $V = \mathbb{V}(f)$ be a curve. The set of nonsingular points of f is an open, dense set in V.

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- The map between the curve $y^2 = x^2 + x^3$ and its blowup is a birational morphism. Why?

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What is the best way to ensure that B has no singularities above the origin?

- Replace the origin with the space of lines passing through it!

Definition

Let $\{f_1, f_2, \ldots, f_n\}$ be polynomials in two variables x and y. Let $V = \mathbb{V}(f_1, f_2, \ldots, f_n)$ be a variety in $\mathbb{C}^2$. The *blowup of V at the origin* is the variety generated by $\{f_1, f_2, \ldots, f_n, y - xw\}$, where w is a new variable representing the slope of any line through the origin.

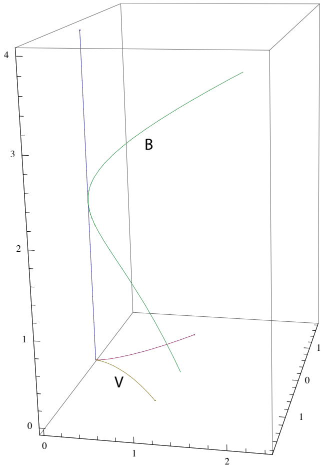
The Blowup of the Curve

$y^2 = x^2 + x^3$

Video courtesy of Hans-Christian v. Bothmer and Oliver Labs.
Source: http://www.calendar.algebraicsurface.net/

Example

Blowup of the Curve $y^2 = x^3$ at the Origin



- Equations: $y^2 = x^3$ and $y = xw$.
- By substitution, $(xw)^2 = x^3$ and $x = w^2$ for all nonzero x. From the first equation, $y = w^3$.
- Parametrization:

$$x \mapsto w^2, \; y \mapsto w^3, \; z \mapsto w.$$

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Theorem (Hironaka's Theorem)

Given any variety W, there is another variety V and a birational morphism $f : V \longrightarrow W$ such V is smooth.

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- We have only been working over the field $\mathbb{C}$, which has characteristic zero. The theorem statement remains an open question over fields of positive characteristic.

Thank You. . .

- to the MAA and the organizers of MathFest 2008 for your time and support
- to Hank, the Notre Dame Mathematics Department, and the NSF for making this project possible
- to Josh, Kaitlyn, and our advisor Adam, for your invaluable suggestions and for all our work together this summer!

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