

# Resolution of Singularities in Algebraic Varieties

Emma Whitten

Summer 2008

## 1 Introduction

Recall that algebraic geometry is the study of objects which are or locally resemble solution sets of polynomial equations.<sup>1</sup> To better understand an object, it is helpful to examine its behavior both at points in affine space and at "infinity." The points in affine space,  $\mathbb{A}^n$ , together with additional points at infinity make up the *projective space*  $\mathbb{P}^n$ . Alternatively,  $\mathbb{P}^n$  can be thought of as the set of lines through the origin in  $\mathbb{A}^{n+1}$ , written as the set of equivalence classes  $\{[x_0, x_1, \dots, x_n]\}$ . Two points  $[x_0, x_1, \dots, x_n]$  and  $[y_0, y_1, \dots, y_n]$  are equivalent if and only if  $[x_0, x_1, \dots, x_n] = [\lambda y_0, \lambda y_1, \dots, \lambda y_n]$  for some scalar  $\lambda$ .

In affine space, we call the common solution set of a collection of polynomials  $\{f_1, f_2, \dots, f_n\}$  an *affine algebraic variety*, denoted  $\mathbb{V}(f_1, f_2, \dots, f_n)$ . In projective space, polynomials must be homogeneous in order to be well-defined on each line through the origin. The vanishing locus of a collection of homogeneous polynomials is a *projective variety*. Given any affine variety  $V \subset \mathbb{A}^n$ , we can take its *projective closure*  $\bar{V} \subset \mathbb{P}^n$ , the smallest projective variety containing it.

While many varieties are smooth, many more have singular points such as nodes or cusps. By studying an object in projective space, it may be possible to view a copy of the object without its singular points. Given a variety  $W \subset \mathbb{A}^n$ , we would like to be able to map  $W$  to a smooth variety  $V$  in the product space  $\mathbb{A}^n \times \mathbb{P}^{n-1}$ , where  $W$  is isomorphic to  $V$  almost everywhere. A useful method for removing singularities from an algebraic variety is called *blowing up*. For example, the curve  $V = \mathbb{V}(y^2 - x^2 - x^3)$  has a singular point at the origin, which can be resolved as in Figure 1.

The curve  $B$  is called the *blowup* of  $V$  at the origin. Notice that  $B$  crosses the  $z$ -axis in exactly two points, corresponding to the two lines tangent to  $V$  at the origin. It may be helpful to think of the blowup  $B$  as being a 3D-parametrization of the original curve, where as  $z = t$  increases, the curve inclines upward and hence never crosses itself at the origin. In this paper we will explain the blowup process and other known methods of removing singularities, ending with a major result in birational geometry proven by Heisuke Hironaka.

---

<sup>1</sup>For an introduction to algebraic geometry, including basic concepts and many of the ideas discussed in this paper, see [4].

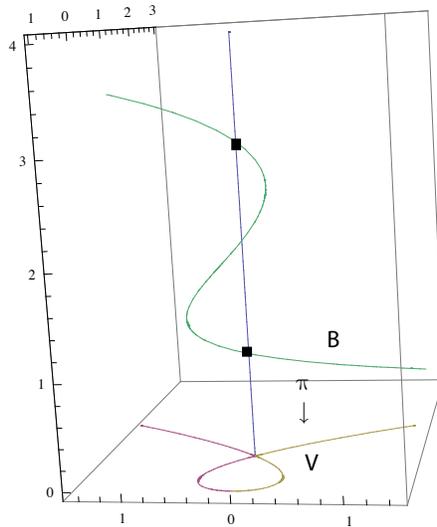


Figure 1: Blowup of  $V$  at the Origin

## 2 Birational Morphisms

The mapping between a variety and its blowup is a type of map called a *birational morphism*, a map similar to an isomorphism. Before we define the map, we first introduce a topology on  $\mathbb{P}^n$ . We will use the new topology to define a more general category of varieties.

**Definition 1.** Let the complements of projective varieties in  $\mathbb{P}^n$  form a collection of open sets. Then they define a topology on  $\mathbb{P}^n$ : the *Zariski topology*.

Note that every projective variety is hence a closed set, while the varieties  $\mathbb{P}^n = \mathbb{V}(0)$  and  $\emptyset = \mathbb{V}(\lambda)$ , where  $\lambda$  is a constant polynomial, are both open and closed.

**Definition 2.** A *quasi-projective variety*  $V$  is a Zariski-locally closed subset of  $\mathbb{P}^n$ , i.e. any intersection of an open set with a closed set.

The set of quasi-projective varieties includes all affine varieties, all projective varieties, and all their Zariski-open subsets. For example, the set  $\mathbb{A}^1 \setminus \{0\}$  is not an affine algebraic variety, but it is an open subset of  $\mathbb{A}^1$  (the complement of the variety  $\mathbb{V}(x, y)$ ) and hence a quasi-projective variety.

**Definition 3.** A morphism of quasi-projective varieties  $\pi : V \rightarrow W$  is called a *birational morphism* if there exists a dense open set  $U \subset V$  such that  $\pi$  is an isomorphism on  $U$ .

**Example 4.** An example of a birational morphism is the blowup map  $\pi$  projecting the curve  $B$  onto  $V = \mathbb{V}(y^2 - x^2 - x^3)$  in Figure 1. There is a clear

bijection between points in the open, dense set  $V \setminus \{0\}$  and their preimages in  $B$ . The preimage of the origin is the set of two points where  $B$  crosses the  $z$ -axis.

### 3 The Blowup of $\mathbb{A}^n$ at a Point $p$

We have just seen an example of a blowup of a variety,  $V$ , at the origin. Let us define the blowup of affine space,  $\mathbb{A}^n$ , at a point  $p$ .

The primary goal while blowing up at  $p$  is to leave  $\mathbb{A}^n$  essentially unchanged at every point except at  $p$ , where we will replace  $p$  with an entire copy of  $\mathbb{P}^{n-1}$  (hence the term blowing up!). Thus every line through the origin in  $\mathbb{A}^n$  (i.e. any possible line tangent to a variety) can be associated with a unique corresponding point in  $\mathbb{P}^{n-1}$ , thereby separating the lines through  $p$  in a very natural way. While blowing up, we can assume that  $p$  is the origin ( $O$ ) by applying an appropriate change of coordinates.

Most points from which we will blow up a variety will be *singularities*:

**Definition 5.** Let  $V = \mathbb{V}(f_1, f_2, \dots, f_m) \subset \mathbb{A}^n$ . A singularity, or singular point, of  $V$  is a point  $p$  such that the Jacobian matrix

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} (p)$$

has rank strictly less than  $\min(m, n)$ .

For a variety of a single polynomial  $\mathbb{V}(f)$ , the definition of a singular point above is equivalent to the requirement that all partial derivatives vanish at a singular point. To begin constructing the blowup at a point, we define the blowup surface:

**Definition 6.** Let the blowup surface  $B$  be the set of all pairs consisting of a point  $q$  in  $\mathbb{A}^n$  and a line from  $q$  through the origin of  $\mathbb{A}^n$ . Note that  $B$  is in the product space  $\mathbb{A}^n \times \mathbb{P}^{n-1}$ . Thus we have

$$B = \left\{ (q, \vec{q}) \mid q \in \mathbb{A}^n \right\} \subset \mathbb{A}^n \times \mathbb{P}^{n-1}.$$

**Proposition 7.** If  $(x_1, x_2, \dots, x_n; y_1 : y_2 : \dots : y_n)$  are coordinates of  $\mathbb{A}^n \times \mathbb{P}^{n-1}$ , then  $B$  is the quasi-projective variety

$$\mathbb{V}(x_i y_j - x_j y_i \mid 0 \leq i < j \leq n) \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}.$$

*Proof.* The point  $x = (x_1, x_2, \dots, x_n) \in \mathbb{A}^n$  lies on the line  $\ell = [y_1 : y_2 : \dots : y_n] \in \mathbb{P}^{n-1}$  if and only if  $((x_1, x_2, \dots, x_n)$  is a multiple of  $((y_1, y_2, \dots, y_n)$ . That is,  $x$  lies on  $\ell$  if and only if the matrix

$$\begin{bmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{bmatrix}$$

has rank  $\leq 1$ . As we have seen before, this matrix has rank  $\leq 1$  if and only if every  $2 \times 2$  minor vanishes, i.e.  $x_i y_j - x_j y_i = 0$  for every  $i, j$ .  $\square$

**Definition 8.** The blowup of  $\mathbb{A}^n$  at  $O$  is the space  $B$  together with the projection map

$$\begin{aligned} \pi : B &\longrightarrow \mathbb{A}^n \\ (q, \vec{q}) &\longmapsto q. \end{aligned}$$

Note that  $\pi$  generates a one-to-one correspondence between a point  $q$  and the ordered pair  $(q, \vec{q})$  for every point  $q \neq O$ . Thus,  $\pi$  is an isomorphism on the open, dense set  $\mathbb{A}^n \setminus \{O\}$ . The preimage of  $O$  is  $\{O\} \times \mathbb{P}^{n-1}$ , which is isomorphic to  $\mathbb{P}^{n-1}$ . Looking back to the picture,  $\pi^{-1}(O)$  consists of all points in the vertical line above  $O$  (in this case the  $z$ -axis). The set  $\pi^{-1}(O)$  is called the *exceptional divisor* of the blowup. Each line through the origin intersects the exceptional divisor at exactly one point, and hence the blown-up curve intersects the exceptional divisor exactly once for every time the original curve passes through the origin.

We can also denote the space  $B$  by  $B_O(\mathbb{A}^n)$ . Note that the defining equations for  $B_O(\mathbb{A}^n)$  are the  $2 \times 2$  minors of the matrix above.

## 4 The Blowup of an Arbitrary Affine Variety

Using the previous blowup space  $B$  and projection map  $\pi$ , we can generalize the blowup of  $\mathbb{A}^n$  to the blowup of an arbitrary affine algebraic variety  $V$ .

**Definition 9.** Given any affine algebraic variety  $V \subset \mathbb{A}^n$  and any point  $p \in V$ , the blowup of  $V$  at  $p$  is  $B_p(V)$ , the Zariski-closure of  $\pi^{-1}(V \setminus \{p\})$  in  $B$ , together with the projection  $\pi$ .

Again,  $\pi$  restricted to the open, dense set  $B_p(V) \setminus \{\pi^{-1}(p)\}$  is an isomorphism onto  $V \setminus \{p\}$ , and thus  $\pi$  is a projective birational morphism.

**Example 10.** Take, for example, the blowup of the curve  $V = \mathbb{V}(y^2 - x^2 - x^3)$  at the origin as above. Note that the two visible lines tangent to  $V$  at  $(0, 0)$  are mapped to distinct points in  $\mathbb{P}^{n-1}$ , and hence the blown-up curve has no singularity at the origin. The defining equations for  $B_p(V)$  are the defining equations for  $B$  together with the defining equation(s) of  $V$ . These will necessarily be the defining equations for the blowup of any affine variety  $V$ .

The defining equations  $y^2 = x^2 + x^3$  and  $xw = yz$  can be used to derive a parametrization of the blowup  $B$  in affine 3-space, as graphed in Figure ???. Since the blowup is a set of points in  $\mathbb{P}^1 \times \mathbb{A}^2$ , we must examine the blowup in an *affine coordinate chart*, the set of points where one coordinate is never zero. Look at the set of points satisfying  $y^2 = x^2 + x^3$ ,  $xw = yz$ , and  $z \neq 0$ . Since dividing each coordinate by  $z$  now makes no difference in projective space, assume  $z = 1$ . Then  $y = xw$ , and we have

$$(xw)^2 = x^2(x + 1),$$

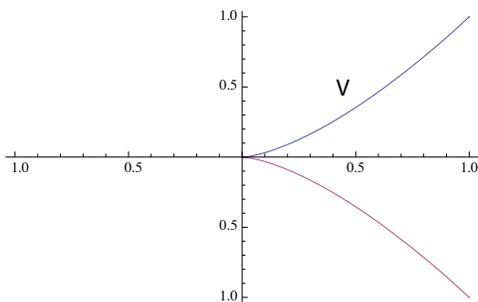


Figure 2: The Cuspidal Cubic:  $V = \mathbb{V}(y^2 - x^3)$

$$x^2(w^2 - x - 1) = 0.$$

For all  $x \neq 0$ ,  $x = w^2 - 1$ . When  $x = w^2 - 1$ ,

$$\begin{aligned} x^2 + x^3 &= (w^2 - 1)^2 + (w^2 - 1)^3 \\ &= w^2(w^2 - 1)^2. \end{aligned}$$

By taking the positive square root we obtain

$$y = w(w^2 - 1).$$

The equations for  $x$  and  $y$  in terms of  $w$  yield a parametrization of  $B$  in affine space:

$$x \mapsto w^2 - 1, \quad y \mapsto w(w^2 - 1), \quad z \mapsto w,$$

which is a smooth curve isomorphic to  $y^2 = x^2 + x^3$  at every point except the origin, as in the curve graphed above.

**Example 11.** Look at the variety defined by the equation for the cuspidal cubic:  $V = \mathbb{V}(y^2 - x^3)$  in Figure 2. The blowup  $B_{(0,0)}(V)$  is defined by the equations  $y^2 = x^3$  and  $xw = yz$ . In the coordinate chart where  $z$  is nonzero,  $y = xw$  and we have that

$$\begin{aligned} (xw)^2 &= x^3, \\ x^2(w^2 - x) &= 0. \end{aligned}$$

Then for all nonzero  $x$ ,  $x = w^2$  and  $y = +\sqrt{(w^2)^3} = w^3$ . Therefore  $B_{(0,0)}(V)$  can be parametrized by the map

$$x \mapsto w^2, \quad y \mapsto w^3, \quad z \mapsto w$$

which is again a smooth curve, birationally equivalent curve in affine 3-space as pictured in Figure 3.

**Example 12.** Blowups can also be used to resolved singular points of a surface, as in the singularity at the origin of the cone  $V = \mathbb{V}(x^2 + y^2 - z^2)$ . The blowup

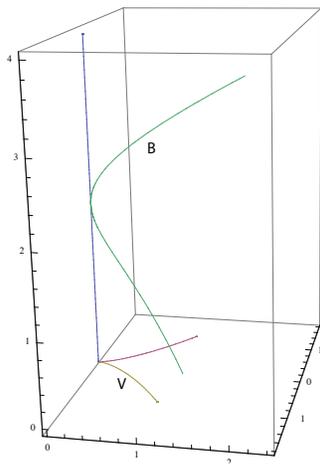


Figure 3: Blowup of  $V$  at the Origin

$B = B_O(V)$  is a subset of  $\mathbb{A}^3 \times \mathbb{P}^2$ . If  $\mathbb{A}^3$  has coordinates  $(x, y, z)$  and  $\mathbb{P}^2$  has coordinates  $[t : u : v]$ , then  $B$  has the following defining equations:

$$\begin{aligned} xu - ty &= 0 & x^2 + y^2 &= z^2 \\ yv - uz &= 0 & t^2 + u^2 &= v^2 \\ xv - ty &= 0. \end{aligned}$$

On the coordinate chart  $v = 1$ , the equations reduce to the following:

$$\begin{aligned} y &= uz \\ x &= ty \\ x^2 + y^2 &= z^2. \end{aligned}$$

Only three equations are necessary, since the remaining two are now superfluous. After eliminating the variables  $x$  and  $y$ , the set of points in  $\mathbb{A}^5$  satisfying the three equations above is isomorphic to the cylinder  $\{(z, t, u) \mid t^2 + u^2 = 1\}$ , which is birationally equivalent to the original cone  $V$ . Note that lines through the origin in  $V$  are mapped to corresponding lines on the cylinder, of which no two intersect.

The cone  $V$  had a finite number of singular points, in fact only one at the origin, but a surface in  $\mathbb{A}^3$  or a higher dimensional space can easily have an infinite number of singular points. For example, the variety  $\mathbb{V}(y^2 - x^3) \subset \mathbb{A}^3$  has a singularity at every point of the form  $(0, 0, z)$ . Blowing up such a variety one point at a time will likely never generate a smooth curve. However, for a curve in  $\mathbb{A}^2$ , blowing up point by point is a successful method:

**Theorem 13.** *For any affine variety  $V \subset \mathbb{A}^2$ , the singularities of  $V$  can be resolved by blowing up the curve at finitely many points.*

The proof requires the following two lemmas:

**Lemma 14.** *An irreducible affine variety  $V \subset \mathbb{A}^2$  has finitely many singular points.*

*Proof of Lemma 14.* Without loss of generality, let  $V = \mathbb{V}(f) \subset \mathbb{A}^2$ . A point  $p \in V$  is singular if both  $\frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$  vanish at  $p$ . If  $f$  has infinitely many singular points, then  $f$  and  $\frac{\partial f}{\partial x}$  are two polynomials which share infinitely many solutions, therefore they must have a common factor. (If  $f$  and  $\frac{\partial f}{\partial x}$  were relatively prime in  $\mathbb{A}^2$ , they would necessarily intersect at  $\deg(f) * \deg(\frac{\partial f}{\partial x})$  points in  $\mathbb{P}^1$ .) Since  $f$  is irreducible,  $f | \frac{\partial f}{\partial x}$ . Likewise  $f | \frac{\partial f}{\partial y}$ . Since  $f$  has larger degree than either partial derivative,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0.$$

Therefore  $f$  must be a constant function with no singular points, contradicting our assumption.  $\square$

**Lemma 15.** *If  $V = \mathbb{V}(f) \subset \mathbb{A}^2$  is an affine variety, the blowup  $B_p(V)$  is nonsingular at  $p$ .*

*Proof of Lemma 15.* The blowup  $B_p(V) = \mathbb{V}(f(x, y), xz - yw)$  is nonsingular at  $p$  if and only if the Jacobean matrix

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & 0 & 0 \\ z & -w & x & -y \end{bmatrix}$$

has rank less than or equal to  $\text{codim } V = 1$ . Since  $p$  is a point in projective space, at least one coordinate of  $p$  must be nonzero, therefore  $B$  is nonsingular at  $p$ .  $\square$

*Proof of Theorem.* Let  $V$  have singular points  $p_1, p_2, \dots, p_m$ . The blowup of  $V$  at  $p_1$ , denoted  $B_{p_1}(V)$  generates a corresponding projective birational map  $\pi_1 : B_{p_1}(V) \rightarrow V$ . The curve  $B_{p_1}(V)$  is necessarily nonsingular at the points  $\pi_1^{-1}(p_1)$ . However, note that since  $\pi_1$  is an isomorphism on  $B_{p_1}(V) \setminus p_1$ , the points  $\pi_1^{-1}(p_2), \dots, \pi_1^{-1}(p_m)$  are singular in  $B_{p_1}(V)$ . Let  $B_{p_1}(V) = B_1(V)$ ,  $B_{\pi_1^{-1}(p_2)}(B_{p_1}(V)) = B_2(V)$ , etc. Then successive blowups yield the following projective birational maps defined on them:

$$\begin{aligned} \pi_2 &: B_{\pi_1^{-1}(p_2)}(B_1(V)) \rightarrow B_1(V) \\ \pi_3 &: B_{\pi_2^{-1}(p_3)}(B_2(V)) \rightarrow B_2(V) \\ &\vdots \\ \pi_m &: B_{\pi_{m-1}^{-1}(p_m)}(B_{m-1}(V)) \rightarrow B_{m-1}(V). \end{aligned}$$

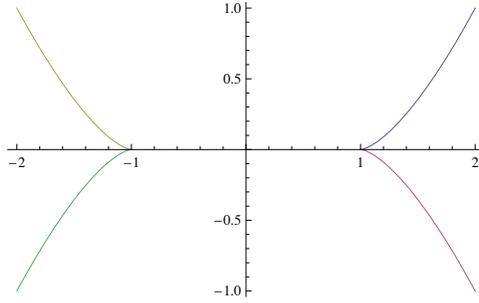


Figure 4:  $\mathbb{V}((y^2 - (x - 1)^3)(y^2 + (x + 1)^3))$

Since the blowups have been chosen to remove singularities at each point  $p_i$ ,  $B_m(V)$  is entirely nonsingular. Finally, the composition of projective maps

$$f = \pi_1 \circ \pi_2 \circ \dots \circ \pi_m : B_m(V) \longrightarrow V$$

is a projective birational morphism on the open, dense set  $B_m(V) \setminus f^{-1}(p_1, p_2, \dots, p_m)$ .  $\square$

**Example 16.** For example, the variety  $V = \mathbb{V}((y^2 - (x - 1)^3)(y^2 + (x + 1)^3))$  in Figure 4 has two singular points, one at  $(-1, 0)$  and the other at  $(1, 0)$ . The blowup  $B_{(1,0)}(V)$  is the union of two curves: one isomorphic to the variety  $\mathbb{V}(y^2 + (x + 1)^3)$ , the other isomorphic to the smooth variety  $\mathbb{A}^1$ . One half of the blowup,  $B_{(1,0)}(\mathbb{V}(y^2 - (x - 1)^3))$ , is pictured in Figure 5. Blowing up  $B_{(1,0)}(V)$  at the second singular point  $(-1, 0)$  yields the union of two new curves which are now each isomorphic to  $\mathbb{A}^1$ .

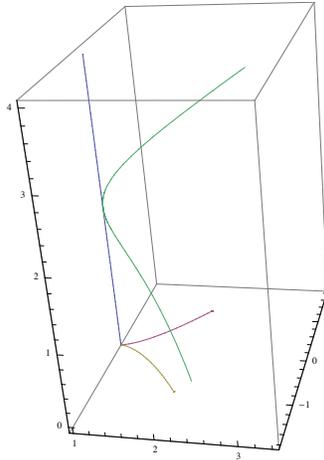


Figure 5: Blowup of  $\mathbb{V}(y^2 - (x - 1)^3)$  at  $(1, 0)$ .

The smooth variety obtained by blowing up a curve at its singular points is a type of *resolution of singularities*.

**Definition 17.** Let  $W$  be a quasi-projective variety. A *resolution of singularities of  $W$*  is a smooth variety  $V$  together with a birational morphism  $\pi : V \rightarrow W$ , where  $\pi$  is an isomorphism above the nonsingular locus of  $W$ .

## 5 Normalization

Blowing up is a very geometric process: no algebra is required beyond simple polynomial equations. Thus it is not surprising that singularities can also be resolved using a purely algebraic method. For any affine variety  $V$ , the process of obtaining a smoother, birationally equivalent variety is called *normalization*.

In order to define a normal affine variety, we first define a normal ring.

**Definition 18.** Let  $R$  be an integral domain with field of fractions  $K$ . Let  $f$  be any monic polynomial in  $R[x]$ . If every root  $a/b \in K$  of  $f$  is also in the original domain  $R$ , then  $R$  is normal.

For example, the domain  $\mathbb{Z}$  is normal.  $\mathbb{Z}$  has field of fractions  $\mathbb{Q}$ , but any rational root of a monic polynomial in  $\mathbb{Z}[x]$  must be an integer. Since  $\mathbb{C}$  is a field,  $\mathbb{C}$  is normal.

Now we can define the related concept of a normal variety:

**Definition 19.** An irreducible affine variety is normal if it has a normal coordinate ring. A reducible affine variety, which can be written as a union of irreducible subvarieties, is normal if each irreducible subvariety is normal. An arbitrary quasi-projective variety is normal if every point is contained in some normal affine variety.

Since  $\mathbb{A}^n$  is a smooth space, ideally it should be considered normal under our definition. Note that the coordinate ring of  $\mathbb{A}^n$ , the polynomial ring  $\mathbb{C}[x_1, x_2, \dots, x_n]$ , is a member of a family of rings called *unique factorization domains*.

**Definition 20.** An integral domain  $R$  is a unique factorization domain (UFD) if every nonzero, nonunit element can be completely factored as a unique product of elements in  $R$ .

**Theorem 21.** *Every unique factorization domain is normal.*

*Proof.* Let  $R$  be a unique factorization domain with field of fractions  $K$ . If  $x \in K$ ,  $x$  can be written uniquely as the irreducible fraction  $a/b$ , where  $a, b \in R$ . Let  $f = x^n + c_{n-1}x^{n-1} + \dots + c_0$  be any monic polynomial in  $R[x]$ . If  $a/b$  is a root of  $f$ , then

$$(a/b)^n + c_{n-1}(a/b)^{n-1} + \dots + c_0 = 0$$

and

$$a^n + c_{n-1}a^{n-1}b + \dots + c_0b^n = 0.$$

Note that  $a^n$  can now be written as a sum of terms, each divisible by  $b$ . Thus  $b|a^n$  and  $a$  and  $b$  must share a factor, contradicting the irreducibility of  $a/b$ .  $\square$

Now since  $\mathbb{C}[x_1, \dots, x_n]$  is a unique factorization domain,  $\mathbb{C}[x_1, \dots, x_n]$  is normal. Hence  $\mathbb{A}^n$  is normal as an affine variety, and  $\mathbb{P}^n$  is a normal as a quasi-projective variety (because it can be covered by  $n$  copies of  $\mathbb{A}^n$ ).

**Example 22.** Not every variety is normal. The curve  $V = \mathbb{V}(y^2 - x^3)$ , the cuspidal cubic, has coordinate ring  $\mathbb{C}[x, y]/(y^2 - x^3) \simeq \mathbb{C}[t^2, t^3] = \mathbb{C}(V)$ . The fraction  $t^3/t^2 = t$  is a root of the polynomial  $x^2 - t^2$  in  $\mathbb{C}(V)[x]$ , but  $t \notin \mathbb{C}(V)$ . Therefore  $\mathbb{C}(V)$  is not normal as a ring, and  $V$  cannot be a normal variety.

Any variety can be made normal by a process called *normalization*. Given any affine algebraic variety  $V$ , we see that in order for  $V$  to be normal, it must have a normal coordinate ring. Fortunately, any commutative ring can be normalized as follows:

**Definition 23.** Let  $R$  be a commutative ring with identity. Then  $R_N$ , the normalization of  $R$ , is the set of all "rational" polynomial solutions in  $R[x]$ , i.e. the elements of the field of fractions of  $R$  which satisfy any monic polynomial with coefficients in  $R$ .

We can now define the normalizations of affine and quasi-projective varieties:

**Definition 24.** Let  $V$  be an irreducible affine variety with coordinate ring  $\mathbb{C}(V)$ . The normalization of  $V$  is the variety  $W$  with coordinate ring  $\mathbb{C}[W] = \mathbb{C}(V)_N$ , together with a birational map  $f : W \rightarrow V$ . If  $V$  is a reducible affine variety, then  $V$  can be written as a union of irreducible subvarieties. The normal variety  $W$  is the union of the normalizations of the irreducible subvarieties.

**Example 25.** For example, the variety  $V = \mathbb{V}(y^2 - x^3)$  has coordinate ring  $\mathbb{C}(V) \simeq \mathbb{C}[t^2, t^3]$ , which is not normal. Note that  $t$  is a root of the equation  $x^2 - t^2$  in  $\mathbb{C}(V)[x]$ . Therefore  $t$ , and every element of  $\mathbb{C}[t]$ , is in the normalization of  $\mathbb{C}[t]$ . Since  $\mathbb{C}[t]$  is a unique factorization domain,  $\mathbb{C}[t]$  is normal. Therefore  $\mathbb{C}[t]$  is the normalization of  $\mathbb{C}[V]$  and  $\mathbb{A}^1$  is the corresponding normal variety, with birational map

$$\begin{aligned} f : \mathbb{A}^1 &\longrightarrow V \\ t &\longmapsto (t^2, t^3). \end{aligned}$$

For an affine variety  $V$ , the normalization of  $V$  clearly exists.  $\mathbb{C}(V)_N$ , the normalization of  $\mathbb{C}(V)$ , is isomorphic to  $\mathbb{C}[x_1, x_2, \dots, x_n]/I$  for some ideal  $I$ , where  $I$  generates the normal variety  $W = \mathbb{V}(I)$ . It is not obvious that  $W$  is affine.<sup>2</sup>

**Definition 26.** Let  $V$  be an arbitrary quasi-projective variety. The normalization of  $V$  is a normal variety  $W$  together with a birational morphism  $f : W \rightarrow V$ .

---

<sup>2</sup>The proof requires a significant amount of abstract algebra. For a proof see page 129 of [3]

$V$  such for any other normal variety  $U$  and dominant morphism  $g : U \rightarrow V$ , there is a unique map  $\phi : U \rightarrow W$  where  $g = f \circ \phi$ .<sup>3</sup>

It is not obvious that the normalization of a quasi-projective variety exists, nor that the normalization of a quasi-projective variety will also be quasi-projective.<sup>4</sup>

In the example above,  $V = \mathbb{V}(y^2 - x^3)$ , normalizing  $V$  effectively removed its singular point at the origin, creating a smooth variety isomorphic to  $\mathbb{A}^1$ . In fact, normalization resolves the singularities of any curve in the affine plane. The proof relies on the following lemma:

**Lemma 27.** *In a normal variety, the set of singular points has codimension  $\geq 2$ .*<sup>5</sup>

By the lemma, the set of singular points of any normal variety in  $V \subset \mathbb{A}^2$  must have dimension 0. That is, the normalization of  $V$  must be nonsingular. We now have the following theorem:

**Theorem 28.** *For an affine variety  $V \subset \mathbb{A}^2$ , the normalization of  $V$  is nonsingular.*

## 6 General Blowups and Hironaka's Theorem

Both blowing up and normalization are effective ways to resolve singularities on affine curves, but blowing up can be generalized to resolve singularities of more general quasi-projective varieties. In order to define the general blowup of a variety, we first need to define rational maps and graphs.

**Definition 29.** Let  $U_1, U_2$  be open, dense subsets of a quasi-projective variety  $V$ . Two morphisms of quasi-projective varieties  $\phi_1 : U_1 \rightarrow W$  and  $\phi_2 : U_2 \rightarrow W$  are equivalent if they agree on the intersection  $U_1 \cap U_2 \subset V$ .

**Definition 30.** An equivalence class of such morphisms is called a rational map, denoted  $\phi : V \dashrightarrow W$ .

Note that a rational map is not actually a map (hence the dashed arrow notation), and the composition of two rational maps is not always well-defined! In fact, the composition of rational maps  $\phi \circ \psi$  is only well-defined when the image of  $\psi$  is dense.

**Definition 31.** If  $\chi : U \rightarrow W$  is any map, the graph of  $\chi$  is the set

$$\Gamma_\chi = \{(x, \chi(x)) | x \in U\} \subset U \times W.$$

---

<sup>3</sup> $g$  is dominant when its image  $g(U)$  is dense in  $V$ .

<sup>4</sup>For a proof, see page 131 of [3]

<sup>5</sup>For a proof, see page 127 of [3]

**Definition 32.** Given a rational map  $\phi : V \dashrightarrow W$  and a representative of the equivalence class  $\chi : U \rightarrow W$ , the graph of  $\phi$  is the closure of the graph of  $\chi$ :

$$\Gamma_\phi = \overline{\{(x, \chi(x)) \mid x \in U\}} \subset U \times W.$$

**Example 33.** For example, we can blow up  $\mathbb{A}^n$  at the origin using our new terminology. Let  $f$  be the map

$$\begin{aligned} \mathbb{A}^n \setminus \{O\} &\longrightarrow \mathbb{P}^{n-1} \\ (x_1, x_2, \dots, x_n) &\longmapsto [x_1 : x_2 : \dots : x_n]. \end{aligned}$$

The graph  $\Gamma_f$  is the set

$$\{(x, f(x))\} \subset \{\mathbb{A}^n \setminus \{O\}\} \times \mathbb{P}^{n-1}.$$

Taking the closure of  $\Gamma_f$  in the space  $\mathbb{A}^n \times \mathbb{P}^{n-1}$  yields the set

$$B = \{(q, \vec{q}) \mid q \in \mathbb{A}^n\} \subset \mathbb{A}^n \times \mathbb{P}^{n-1},$$

our blowup space in the original example. Now we can identify the original blowup  $B_O(\mathbb{A}^n)$  with the graph of the rational map

$$\begin{aligned} \mathbb{A}^n &\dashrightarrow \mathbb{P}^{n-1} \\ (x_1, x_2, \dots, x_n) &\longmapsto [x_1 : x_2 : \dots : x_n]. \end{aligned}$$

**Definition 34.** Let  $V = \mathbb{V}(f_1, f_2, \dots, f_n)$  be a quasi-projective variety in  $\mathbb{P}^n$ , where  $f_1, f_2, \dots, f_n$  are homogeneous polynomials. Let  $f$  be the rational map

$$\begin{aligned} f : V &\dashrightarrow \mathbb{P}^{n-1} \\ x &\longmapsto [f_1(x) : f_2(x) : \dots : f_n(x)]. \end{aligned}$$

The graph  $\Gamma_f$  of the map  $f$  and the projection map  $\pi : \Gamma_f \rightarrow V$  make up the blowup of  $V$  along the ideal  $I = (f_1, f_2, \dots, f_n)$ , denoted  $B_I(V)$ .

In fact, for any quasi-projective variety  $V$ , there is some ideal  $I$  such that  $B_I(V)$  is nonsingular.

**Theorem 35** (Hironaka's Theorem). *Let  $V$  be an arbitrary quasi-projective variety. Then there exist polynomials  $f_1, f_2, \dots, f_n$  such that the graph  $\Gamma_\phi$  of the rational map*

$$\begin{aligned} \phi : V &\dashrightarrow \mathbb{P}^{n-1} \\ x &\longmapsto [f_1(x) : f_2(x) : \dots : f_n(x)] \end{aligned}$$

*(together with its projection onto  $V$ ) is a resolution of singularities of  $V$ .*

The proof of Hironaka's Theorem is complex, and as much research is being done toward understanding and refining the proof as toward generating new theorems. Rather than prove the theorem for quasi-projective varieties, Hironaka proves a more general statement resolving the singularities of *ideal sheaves*. What is an ideal sheaf?

An ideal sheaf is a subset of the *structure sheaf* of a variety. If  $V$  is a variety with coordinate ring  $\mathbb{C}[V]$ , and  $U$  is an open subset of  $V$  containing  $x$ , let  $\mathcal{O}_x$  be the ring of rational functions  $f/g$  with  $f, g \in \mathbb{C}(U)$ . Let  $\mathcal{O}_x(U)$  be the intersection of all  $\mathcal{O}_x$  with  $x \in U$ , i.e.

$$\mathcal{O}_V(U) = \bigcap_{x \in U} \mathcal{O}_x.$$

Then the structure sheaf of  $V$  is the relation associating each open set  $U \subset V$  with the ring  $\mathcal{O}_V(U)$ .

Now let  $W \subset V$  be a closed subset of  $V$ . For every open set  $U \subset V$ , there is a set of polynomials which vanish on the intersection of  $W$  and  $U$ ,

$$\mathcal{I}_W(U) = \{f \in \mathcal{O}_V(U) \mid f(W \cap U) = 0\}.$$

An *ideal sheaf* associates every open subset  $U$  with the set  $\mathcal{I}_W(U)$ . The concept corresponding to a nonsingular variety is a *monomial ideal sheaf*, an ideal sheaf locally generated by a single monomial. Thus the goal of the proof of Hironaka's Theorem is to transform an ideal sheaf into a locally monomial one via successive blowups.

For Hironaka's proof, see [1]. For an accessible paper explaining the key ideas of the proof, see lecture notes on the theorem written by János Kollár, [2]. Hironaka's Theorem is a key result in modern algebraic geometry. A current problem related to the theorem is the attempt to classify all quasi-projective varieties (up to birational equivalence). Hironaka's Theorem has reduced this problem to the much smaller attempt to classify all smooth projective varieties.

## References

- [1] Hironaka, Heisuke. *Resolution of singularities of an algebraic variety over a field of characteristic zero*. I, II. Ann. of Math. **70** 1964, 109-203; **79** 1964, 205-326.
- [2] Kollár, János. *Resolution of Singularities – Seattle Lectures*. 2008.
- [3] Shafarevich, Igor R. *Basic Algebraic Geometry*. Springer-Verlag, 1994.
- [4] Smith, Karen E. *An Invitation to Algebraic Geometry*. Springer-Verlag, 2000.