An Exploration of the Betti Numbers of Modules, Particularly of Codimension 5, Over a Polynomial Ring

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Abstract

This project explores conjectures on the betti numbers of modules over a polynomial ring, with special attention given to the case where the codimension, c, of the module is 5. After giving an overview of relevant definitions and theorems, we introduce 3 main conjectures, including the 1977 Buchsbaum-Eisenbud-Horrocks conjecture and the Total Rank Conjecture. These conjectures are then explored through examples, explanations, and proofs, aided by the computer software Macaulay2. While these conjectures have been proven for smaller values of c, whether or not they hold when $c \geq 5$ is an open question. This project offers specific quasi-examples which appear to violate the conjectures, but then applies known techniques to prove that such modules cannot exist. This exploration builds upon prior research and introduces undergraduate students to new ideas in Commutative Algebra.

1 Basics

Mathematics is essential to nearly every facet of our lives. It is embedded in the intricate details of nature and forms the foundation of our very universe. Math answers some of humanity's biggest questions, but raises just as many. This project investigates a few of the unanswered questions in the field of Commutative Algebra. My research was primarily focused on one of the most basic algebraic objects: polynomials. Although polynomials may seem simple or elementary, they possess many subtle and interesting properties. I was particularily interested in finding relationships between polynomials called syzygies, and determining the betti numbers of sets of polynomials, which count the numbers of syzygies.

This paper will start with defining the basic terminology which will be used throughout before moving to explanations and examples of ideals, varieties, and syzygies. Then we will dive into the crux of my research on the betti numbers and explorations of psuedo-counterexamples of a few major conjectures, which we will show cannot exist.

1.1 Polynomials and Rings

We will follow the treatment in [3] throughout this paper.

First, we must put forth a definition of a polynomial – a foundational concept we've likely all had some past experience with.

Definition 1.1. A polynomial is an algebraic expression consisting of one or more algebraic terms. These terms can involve any number of variables but are subject to a few constraints, which are as follows:

- No variables may be present in the denominator of a fraction;
- No variables may have negative or fractional exponents;
- No square roots of variables are allowed.

Some examples of polynomials include:

- $x^3 + xy^4 - \frac{1}{8}y^2z^3$
- $wx - z^3xw + 7y^6$
- $14z^5 + 7x + 5$

While expressions such as:

- $\sqrt{x} + 7y^3$
- $z^{-3} + 4xy$
- $\frac{3}{x^5} - 9xy^2 + 2$

are not polynomials.

Next we will define the concept of a polynomial ring.

Definition 1.2. The notation $k[x, y, z]$ means the set of all polynomials in variables x, y, z with coefficients in the set k. Very often k will be the set of real numbers, but it could also be the integers or perhaps the complex numbers. In some examples throughout this paper we will use different sets of variables such as $\{x_1, x_2, ..., x_n\}$, which makes it easy to see the number of variables in the ring.

My research focused specifically on sets of polynomials, called ideals, which have special properties.

1.2 What is an Ideal?

In this subsection we will define, and give properties of, ideals. Before we see the formal definition though, let's form an intuitive understanding of what an ideal is. The following analogies and comments will be particularily helpful to readers who have taken a Linear Algebra course.

- An ideal generated by the polynomials $f_1, f_2, \ldots, f_s$ can be thought of as consisting of all "polynomial consequences" of the equations

$$f_1 = f_2 = \ldots = f_s = 0$$

(all linear combinations of the functions in the polynomial ring);

- The ideal generated by a set of polynomials is analogous to the span of finite number of vectors in a vector space;

- The number of minimal generators of any basis of an ideal is an invariant. We will later see that this number is equal to β_1;

- We can change the basis of an ideal without affecting the variety, a geometric object which we will define in a future section.

Now for the formal definition.

Definition 1.3. [3] A subset $I \subset k[x_1, ..., x_n]$ *is an* **ideal** *if it satisfies:*

(i) $0 \in I$,

(ii) *If* $f, g \in I$, *then* $f + g \in I$,

(iii) *If* $f \in I$ *and* $h \in k[x_1, ..., x_n]$, *then* $h \cdot f \in I$. We will call this the "super-multiplicative" property of I.

Definition 1.4. The dimension of an ideal is defined as the dimension of the corresponding geometric object, called a variety, determined by the ideal.

Definition 1.5. The codimension of an ideal c is defined as the ambient dimension n minus the dimension d of the geometric object, where ambient dimension is equal to the number of variables in the polynomial ring. So $c = n - d$. In a later section we will see that there are famous conjectures in Commutative Algebra that relate the codimension to what are called the betti numbers, which we will define in section 2.

Let's now look at an example where we check the conditions to verify whether a set is an ideal or not.

Example 1.6. Let $I = \{f \in \mathbb{R}[x, y, z] | f(2, 0, 1) = 0\}$. Is I an ideal? First, notice that the polynomial $f_1 = x + z - 3$ is in I, since when we plug in (2, 0, 1) to f_1 we see that $f_1(2, 0, 1) = 2 + 1 - 3 = 0$. $f_2 = x^2 + y - 4z$ is another polynomial that is in I, while $f_3 = x^3 + y^3 + z^3$ is not in I. Let's check that the sum $f_1 + f_2$ is still in I.

$$f_1 + f_2 = x + z - 3 + x^2 + y - 4z$$

so $(f_1 + f_2)(2, 0, 1) = 2 + 1 - 3 + 2^2 + 0 - 4$ which is 0. So by definition $f_1 + f_2$ is also in I. We can also show that the product $f_1 \cdot f_2$ is in I. Since both $f_1(2, 0, 1)$ and $f_2(2, 0, 1)$ are equal to 0, their product is equal to $0 \cdot 0 = 0$, which we have already shown is in I.

The following proof shows that I satisfies the necessary closure properties for any arbitrary polynomials in I, thus proving I is an ideal.

Proof. To show that I is closed under addition, let $g, h \in I$. That is

$$g(2, 0, 1) = 0$$

and

$$h(2, 0, 1) = 0.$$

We want to show that $g + h \in I$. Since $g(2, 0, 1) = 0$ and $h(2, 0, 1) = 0$, we can add these equations together to get $(g + h)(2, 0, 1) = 0$. Now since $(g + h)(2, 0, 1) = 0$ and $0 \in I$, we know that $g + h \in I$. Finally we must show that I is super-multiplicative. To do this we let $g \in I$ and let $p \in \mathbb{R}[x, y, z]$ be an arbitrary polynomial. We want to show $p \cdot g \in I$. Since $g \in I$, we know $g(2, 0, 1) = 0$. Then $(p \cdot g)(2, 0, 1) = p(2, 0, 1) \cdot 0 = p \cdot 0 = 0$ and $0 \in I$. So, since I satisfies all 3 conditions, it is an ideal. $\square$

Example 1.7. It is true that ideals can be the same, but in disguise! For instance, we will show that

$$(x - y, x + y) = (x, y).$$

By the notation $(f_1, f_2, ..., f_s)$ we mean the ideal generated by $f_1, f_2, ..., f_s$. If $f_1, f_2, ..., f_s \in k[x_1, x_2, ..., x_n]$, then $(f_1, f_2, ..., f_s)$ is an ideal of $k[x_1, x_2, ..., x_n]$. [3]

Proof. It is clear that each generator $x - y$ or $x + y$ on the left is a combination of generators on the right.

On the other hand, we can get

$$x = \frac{1}{2}(x-y) + \frac{1}{2}(x+y)$$

$$y = \frac{-1}{2}(x-y) + \frac{1}{2}(x+y)$$

So each of x, y can also be written as combinations of the generators on the left. Thus these ideals have the same set of minimal generators and are therefore equal. □

Here's another cool example:

Example 1.8. Is x^3 in the ideal $I = (y^2, xy + x^2)$? The ideal I consists of all combinations

$$fy^2 + g(xy + x^2)$$

where f, g are polynomials. If we let $f = x$ and $g = x - y$ then plugging these in gives

$$(x)y^2 + (x-y)(xy + x^2) = x^3$$

so x^3 is indeed $\in I$. This example illustrates how complicated it can be to tell if a polynomial is in an ideal, since it is not always immediately apparent that multi-term polynomials will foil out to yield a monomial.

The following theorem, which asserts that the sum of two ideals is itself an ideal, is almost like a generalization of the closure under addition property within an ideal.

Definition 1.9. Let I, J be two ideals in a ring $R = k[x_1, \ldots, x_n]$. Then

$$I + J = \{a + b : a \in I \text{ and } b \in J\}.$$

Theorem 1.10. *Let $R = k[x_1, \ldots, x_n]$. If I is an ideal in R and J is an ideal in R, then the set $I + J$ is also an ideal.*

Proof. Suppose I and J are ideals in $k[x_1, ..., x_n]$. We will show that $I + J$ is also and ideal in $k[x_1, ..., x_n]$.

First, $0 \in I$ and $0 \in J$ since both are ideals, so $0 + 0 = 0 \in I + J$. Next, suppose $a_1, a_2 \in I + J$. Then there exist $b_1, b_2 \in I$ and $c_1, c_2 \in J$ such that $a_1 = b_1 + c_1$ and $a_2 = b_2 + c_2$ by the definition of $I + J$.

Now $a_1 + a_2 = b_1 + c_1 + b_2 + c_2 = (b_1 + b_2) + (c_1 + c_2)$, and since I and J are ideals, $b_1 + b_2 \in I$ and $c_1 + c_2 \in J$, so $a_1 + a_2 \in I + J$.

Finally, to check closure under multiplication, let $a \in I + J$ and let $f \in k[x_1, ..., x_n]$ be any polynomial. Then, as before, there exists $b \in I$ and $c \in J$ such that $a = b + c$.

Notice that $f \cdot a = f \cdot (b + c) = (f \cdot b) + (f \cdot c)$ where $f \cdot b \in I$ and $f \cdot c \in J$ since I and J are ideals.

Thus, $I \cdot a \in I + J$, so $I + J$ is closed under multiplication by polynomials in the ring. So $I + J$ satisfies all 3 necessary properties and this completes the proof that $I + J$ is an ideal. □

Now that we've proved Theorem 1.10 and have a better understanding of ideals, we can define varieties.

1.3 What is a Variety?

Definition 1.11. [3] *Let k be a field, and let $f_1, ..., f_s$ be polynomials in $k[x_1, ..., x_n]$. Then*

$$\mathbf{V}(f_1, ..., f_s) = \{(a_1, ..., a_n) \in k^n : f_i(a_1, ..., a_n) = 0 \text{ for all } 1 \leq i \leq s\}. \tag{1.1}$$

We call $\mathbf{V}(f_1, ..., f_s)$ the **affine variety** *defined by $f_1, ..., f_s$.*

In other words, a variety is essentially the vanishing locus of a bunch of polynomials.

Example 1.12. Say we are working in $\mathbb{R}[x, y, z]$. Consider the set

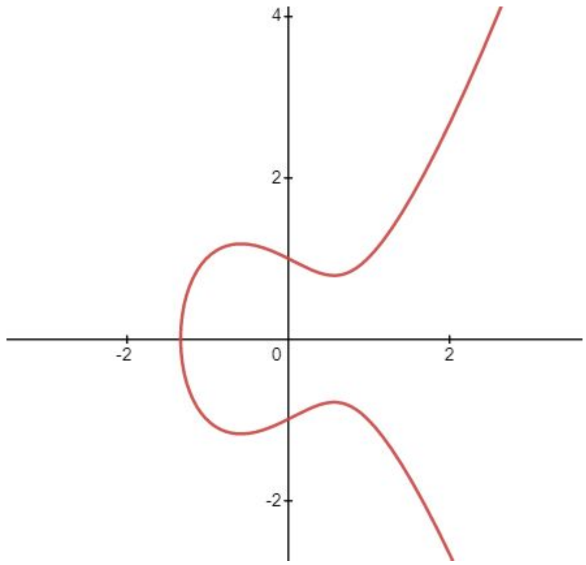
$$X = V(x - 2, y - 3, z - 4).$$

The set of equations $\{x - 2 = 0, y - 3 = 0, z - 4 = 0\}$ define the intersection of 3 planes in $\mathbb{R}^3$ which intersect at a point. Thus, X is the subset of $\mathbb{R}^3$ consisting of one point:

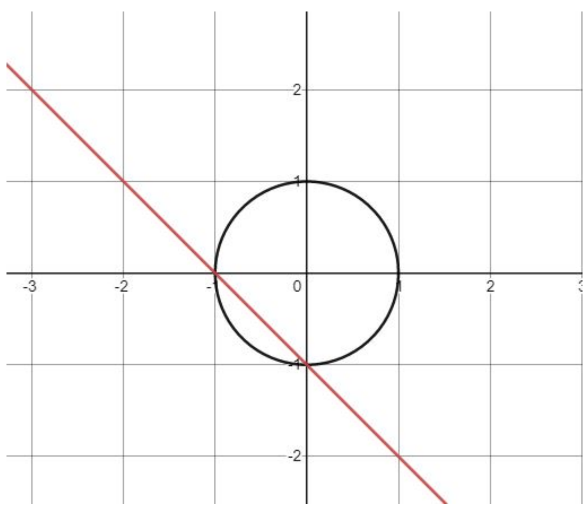
$$X = V(x - 2, y - 3, z - 4) = \{(2, 3, 4)\}$$

since plugging in (2,3,4) is the only way to get zero for all three polynomials.

When learning about varieties it can be helpful to see visual, graphical represntations of the varieties of both individual polynomials and sets of polynomials. For example, the affine variety of the polynomial $y^2 - (x^3 - x + 1) = 0$ (or any single polynomial) will just be the graph of the polynomial itself, as seen below.



On the other hand, the affine variety of a set of polynomials is the set of their intersection points. The polynomials $y + x + 1 = 0$ and $x^2 + y^2 = 1$ has affine variety $V = (-1, 0), (0, -1)$ which is the set of the 2 points of intersection between the graphs.



Many subsets of $\mathbb{R}^n$ are affine varieties. For example:

Example 1.13. Any point $(a_1, \ldots, a_n) \in \mathbb{R}^n$ will be a variety since:

$$\{(a_1, \ldots, a_n)\} = V(x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n).$$

1.4 Does a Variety Depend on a Set or an Ideal?

In the previous section we defined varieties as subsets of $\mathbb{R}^n$ which are the vanishing loci of sets of polynomials. Based on this definition, a variety seems to depend upon the set of polynomials it is defined by. In this subsection we will consider an alternate possibility, namely whether a variety may instead depend upon an ideal. Before we determine whether this is the case we will first show how a variety can give rise to an ideal.

Definition 1.14. Let X be a subset of k^n. And $R = k[x_1, \ldots, x_n]$. Then the ideal I of X is defined as:

$$I(X) = \{f \in R | f(a) = 0 \text{ for all } a \in X\}.$$

Proposition 1.15. *If X is a set, then $I(X)$ is always an ideal of R.*

Proof. To prove that $I(X)$ is an ideal, we must show that it satisfies all 3 properties identified in the definition of an ideal. For starters it is obvious that $0 \in I(X)$ since the zero polynomial vanishes on all of k^n, and so, in particular it vanishes on X. Next, suppose that $f, g \in I(X)$ and $h \in k[x_1, x_2, ..., x_n]$. Let $(a_1, a_2, ..., a_n)$ be an arbitrary point of X. Then $f(a_1, a_2, ..., a_n) + g(a_1, a_2, ..., a_n) = 0 + 0 = 0$, $h(a_1, a_2, ..., a_n)f(a_1, a_2, ..., a_n) = h(a_1, a_2, ..., a_n) \cdot 0 = 0$, and it follows that $I(X)$ is an ideal, since we have shown closure under addition and polynomial multiplication. □

The following are some important and interesting properties of ideals, some of which we will not prove.

- If we have an affine variety, V, which is a subset of k^n, then $I(V)$, a subset of the polynomial ring, is the ideal of V;
- The ideal generated from the polynomials in the ring which vanish on the set V is a subset of, but not necessarily equal to, the ideal of the variety of said functions;
- If V, W are affine varieties then:
  - (i) V is a subset of W if and only if $I(W)$ is a subset of $I(V)$
  - (ii) $V = W$ if and only if $I(V) = I(W)$.

The following example will answer the question posed by the title of this subsection.

Example 1.16. If we have some points where

$$f_1 = 0, f_2 = 0, \ldots, f_n = 0,$$

Then surely the polynomial f_1+f_2 will also be zero on this set. Then it follows that the polynomial $f_1+x^3f_2$ will also be zero on this set.

So actually **any** polynomial that's in the **ideal** generated by the f_i will vanish on this set X. Thus, we can conclude that a variety depends not on a set of polynomials but on the ideal itself.

Here are some other useful and interesting facts about varieties, which we will not prove.

- Conic sections, graphs of polynomial functions, and graphs of rational functions are all affine varieties;
- Affine varieties can be the empty set;
- The intersection and union of two affine varieties is also an affine variety. This property is related to the fact that the sum and product of 2 ideals is again an ideal.

What happens when we let $X = V(I)$? In other words:

Question 1.17. If I is an ideal, is $I(V(I)) = I$?

To prove two sets are equal, we must show that the are subsets of eachother. In this case, however, $I \subset I(V(I))$ holds, but the other inclusion can fail. For example:

Example 1.18. Take $I = x^5$, in $\mathbb{R}[x]$. Then $V(I) = \{0\}$, since the only solution to $x^5 = 0$ is $x = 0$. Now we will calculate $I(V(I))$ and see if it is equal to I. Starting with $V(I) = V(x^5) = \{0\}$ so we have $I(0) = (x)$ (i.e. the ideal of all multiples of x) since we get 0 when we multiply $x = 0$ by any number. Thus I and $I(V(I))$ is not a subset of I so these are actually two different ideals.

1.5 Syzygies

Building upon our newfound understanding of ideals, we define another closely related concept.

Definition 1.19. [6] Let I be an ideal, defined by the minimal generators $f_1, f_2, \ldots, f_m$, in R. That is, $I = (f_1, ..., f_m)$ in $R = k[x_1, ..., x_n]$ Then a **first syzygy** of I is a vector $(c_1, ..., c_m)^T$ of polynomials such that

$$c_1f_1 + ... + c_mf_m = 0. \tag{1.2}$$

If $\mathbf{v}_1, ..., \mathbf{v}_t$ are first syzgies, a second syzygy is a vector $\mathbf{w} = (d_1, ..., d_t)$ such that

$$\sum_{i=1}^{t} d_i\mathbf{v}_i = 0. \tag{1.3}$$

A third syzygy is a relation on the second syzygies in the same way and so on.

Example 1.20. Take the ideal $I = (x, y, z)$. Since there are no commonalities or 'overlap' between x,y, and z, we can clearly see that this ideal has 3 minimal generators. In this case,

$$(z)(x) + (0)(y) - (x)(z) = 0$$

is an example of a minimal syzygy. A non-minimal syzygy is one which is a linear combination of other syzygies, and therefore contains polynomial terms of degree greater than 1. See the below figure for the other minimal and non-minimal syzygies.

Generators	x	y	z
Minimal Syzygy	$-y$	x	0
Minimal Syzygy	z	0	$-x$
Minimal Syzygy	0	z	$-y$
Non-Minimal Syzygy	xy	$-x^2$	0
Non-Minimal Syzygy	yz	0	$-yx$
Non-Minimal Syzygy	yz^2	0	$-zyx$

Figure 1: Minimal vs. Non-minimal Example

Thus we have 3 minimal (1st syzygies), which are the 3 listed above.

Is there any combination of these 3 syzygies that give us the 0 vector? In other words, do there exist polynomials A, B, C such that multiplying the minimal syzygies by these polynomials as shown below results in all terms cancelling out?

$$A(-y, x, 0)^T + B(z, 0, -x)^T + C(0, z, -y)^T$$

Here's a solution:

$$z(-y, x, 0)^T + y(z, 0, -x)^T - x(0, z, -y)^T.$$

This tuple $(z, y, -x)$ is therefore a second syzygy. Thus, $I = (x, y, z)$ has three generators, three (minimal) 1st syzygies, one 2nd (nontrivial) syzygy.

Example 1.21. Now let's see an example of an ideal which does not have any higher syzygies.
Let $J = (x^2, xy, y^2)$ be our ideal. Then the minimal syzygies of J are those seen in Figure 2, on the following page. In this example the only 2nd syzygy is the trivial one, so $J = (x^2, xy, y^2)$ has three minimal generators, two 1st syzygies, and no higher (nontrivial) syzygies.

Generators	x^2	xy	y^2
Minimal Syzygy	0	y	$-x$
Minimal Syzygy	$-y$	x	0

Figure 2: Trivial 2nd Syzygy Example

In our final syzygy example we will look at another way to express syzygies.

Example 1.22. [6] For the ideal $I = (x^2, xy, y^2)$, we have:

$$y(x^2) - x(xy) + 0(y^2) = x^2y - x^2y = 0 \tag{1.4}$$

$$y^2(x^2) + 0(xy) - x^2(y^2) = x^2y^2 - x^2y^2 = 0 \tag{1.5}$$

$$0(x^2) + y(xy) - x(y^2) = xy^2 - xy^2 = 0. \tag{1.6}$$

We can express these syzygies as vectors.

$$\mathbf{v}_1 = \begin{pmatrix} y \\ -x \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} y^2 \\ 0 \\ -x^2 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ y \\ -x \end{pmatrix}. \tag{1.7}$$

We can see that $\mathbf{v}_2 = y\mathbf{v}_1 + x\mathbf{v}_3$ so this syzygy is in the span of $\mathbf{v}_1$ and $\mathbf{v}_3$. In other words, $\mathbf{v}_2$ is a non-minimal syzygy generated by $\mathbf{v}_1$ and $\mathbf{v}_3$. In fact, $\mathbf{v}_1$ and $\mathbf{v}_3$ generate all first syzygies of I. As in the previous example, there are no second syzygies in this case. We can see this by supposing

$$a\mathbf{v}_1 + b\mathbf{v}_3 = a\begin{pmatrix} y \\ -x \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ y \\ -x \end{pmatrix} = 0 \tag{1.8}$$

for a and b. But then $ay = -bx = 0$ so $a = b = 0$, so there are no relations between the first syzygies.

2 Betti Numbers and Examples

In this section we will define the betti numbers, explore some of their properties, and see how they can be calculated from a sequence of numbers or a set of polynomials.

Definition 2.1. Let I be an ideal. We can associate to I a set of numbers, called the betti numbers, which are defined as follows:

- $\beta_0 = 1$;
- $\beta_1 =$ the number of minimal generators for the ideal;
- $\beta_2 =$ the number of minimal 1st syzygies for the ideal;
- $\beta_3 =$ the number of minimal 2nd syzygies for the ideal;

... the pattern continues.

Definition 2.2. The *betti diagram* of a graded free module M is the array given by the following table, where the i, j-*th* entry is $\beta_{i,i+j}$.

$$\beta(M) = \begin{array}{|llll|} F_0 & F_1 & ... & F_r \\ \hline \beta_{0,0}(M) & \beta_{1,1}(M) & ... & \beta_{r,r}(M) \\ \beta_{0,1}(M) & \beta_{1,2}(M) & ... & \beta_{r,r+1}(M) \\ \beta_{0,2}(M) & \beta_{1,3}(M) & ... & \beta_{r,r+2}(M) \\ \vdots & \vdots & ... & \vdots \end{array}$$

Definition 2.3. [6] We say that a free resolution P is *pure* if its betti diagram has only one entry per column. In other words, if we can write:

$$\mathbf{P} : 0 \longrightarrow R(-d_r)^{\beta_r} \xrightarrow{\delta_r} R(-d_{r-1})^{\beta_{r-1}} \xrightarrow{\delta_{r-1}} ... \xrightarrow{\delta_1} R(-d_0)^{\beta_0} \xrightarrow{\delta_0} M \longrightarrow 0. \tag{2.1}$$

We call $d = \{d_0, d_1, ..., d_r\}$ the degree sequence of P.

Example 2.4. [6] Let $R = \mathbb{R}[x, y]$, $I = (x^2, xy, y^2)$, and $M = R/I$ as in Example 1.22. The graded version of the free resolution we found before is

$$0 \longrightarrow R(-3)^2 \xrightarrow{\begin{pmatrix} x & 0 \\ -y & x \\ 0 & -y \end{pmatrix}} R(-2)^3 \xrightarrow{\begin{pmatrix} x^2 & xy & y^2 \end{pmatrix}} R \longrightarrow M \longrightarrow 0 \tag{2.2}$$

Since there is one generator for $F_0 = R$ of degree 0, we have $\beta_{0,0} = 1$. Likewise, there are 3 generators for F_1 of degree 2 and 2 generators for F_2 of degree 3 so $\beta_{1,2} = 3$ and $\beta_{2,3} = 2$. Which gives the betti table below, where all the dashes represent zeros.

$$\beta(M) = \begin{array}{|ccc} F_0 & F_1 & F_2 \\ \hline 1 & - & - \\ - & 3 & 2 \\ - & - & - \end{array}$$

Note that this resolution is pure with degree sequence $d = \{0, 2, 3\}$.

Example 2.5. [6] Let $R = k[x, y]$, $I = (x^3, x^2y, y^2)$, and $M = R/I$ as in Example 3.2 We have:

$$\beta(M) = \begin{array}{|ccc} F_0 & F_1 & F_2 \\ \hline 1 & - & - \\ - & 1 & - \\ - & 2 & 2 \\ - & - & - \end{array}.$$

Note that this resolution is not pure since the middle column of our betti table contains two entries.

3 Three Big Conjectures

In this section we will state the results in terms of *modules*, which are generalizations of the notions of ideals. A module, for instance, could be defined by the polynomials in an ideal. The goal of this section is to introduce 3 major conjectures on betti numbers and their sums.

Conjecture 3.1. [2] (Buchsbaum-Eisenbud, Horrocks (B-E-H) 1977) Suppose that M is a module of codimension c. Then

$$\beta_i(M) \geq \binom{c}{i}.$$

Note that for instance when $c = 5$, B-E-H says: $\beta_0 \geq 1$, $\beta_1 \geq 5$, $\beta_2 \geq 10$, $\beta_3 \geq 10$, $\beta_4 \geq 5$, $\beta_5 \geq 1$.

Conjecture 3.2. Suppose that M is a module of codimension c. Then

$$\sum_{i=0}^{c} \beta_i(M) \geq 2^c.$$

Since this sum will be referred to frequently in this section, we will use the notation $\beta(M) := \sum \beta_i(M)$.

Conjecture 3.3. Suppose that M is a module of codimension c. Then

$$\text{If } \beta(M) > 2^c, \text{ then } \beta(M) \geq 2^c + 2^{c-1}.$$

In this section we are going to investigate whether the preceding three conjectures hold. These conjectures are significant and useful since they allow us to find the smallest possible betti numbers of an ideal as well as the smallest possible betti sum just by knowing what the codimension is. Here is a simple example for $c = 5$.

Example 3.4. Let

$$d = 0, c = 5 : I = (a^2, b^2, c^2, d^2, e^2).$$

Then our object is a single point, and we can calculate the betti numbers and see that they are $\{1, 5, 10, 10, 5, 1\}$, so B-E-H holds.

Example 3.5. Now let's try an example with a smaller value of c to get a feel for the conjectures and how betti numbers are calculated. If we let $c = 3$ then, since the conjectures are true when $c = 3$, we know that the betti numbers are at least $\{1, 3, 3, 1\}$; their sum is at least 8; and if their sum is greater than 8 then it is at least $8 + 4 = 12$.

This shows that even if $c = 3$ these betti numbers are adding to at least 12, which is fairly large. For $c \leq 4$ all 3 conjectures are known, but for $c \geq 5$ only conjecture 2 is known and the others are open. Herzog and Kuhl showed that if M is a pure module with degree sequence $\{d_0, ..., d_c\}$, then for all $i \geq 1$ we have:

$$\beta_i(M) = \beta_0(M) \prod_{j \neq i} \frac{d_j}{|d_i - d_j|}.$$

We will refer to this formula informally as the 'Π product formula,' or simply the 'Π formula.' Note that the codimension c is the length of this sequence of d_i's, keeping in mind that we start with d_0. Here's an interesting observation: When our degree sequence input is $\{0, d_1, d_2\}$ (i.e. $c = 2$), performing the calculation by hand (using the Π product formula) will yield output that will be of the form $\{|d_1 - d_2|, d_2, d_1\}$. We will frequently need to refer to the last factors of the formula above, so we define $\pi_i(d) = \beta_i/\beta_0$. In other words,

$$\pi_i(d) = \prod_{j \neq i} \frac{d_j}{|d_i - d_j|}.$$

Proof. By appying the Π the formula we can show this to be true.

$$\pi_0(d) = 1 \tag{3.1}$$

$$\pi_1(d) = \frac{d_1 d_2}{|d_2 - d_1||d_0 - d_1|} = \frac{d_1 d_2}{|d_2 - d_1|d_1} = \frac{d_2}{|d_2 - d_1|} \tag{3.2}$$

$$\pi_2(d) = \frac{d_1 d_2}{|d_2 - d_1||d_0 - d_1|} = \frac{d_1 d_2}{|d_1 - d_2|d_2} = \frac{d_1}{|d_1 - d_2|} \tag{3.3}$$

Then we have $\{1, \frac{d_2}{|d_2-d_1|}, \frac{d_1}{|d_1-d_2|}\}$ as our output, and clearing the denominators to avoid fractions gives us $\{|d_2 - d_1|, d_2, d_1\}$ as expected. □

Proposition 3.6. *Multiplying any degree sequence* $\{d_0, d_1, ..., d_n\}$ *by any* $a \in \mathbb{Z}$ *will not affect the numbers output by the* Π *formula.*

Proof.

$$\text{Suppose that our input is the degree sequence: d} = \{0, m_1, m_2, m_3, ..., m_c\}. \tag{3.4}$$

Then we would get an output from the formula as:

$$\pi_0(d) = \frac{m_1 \cdot m_2 \cdot m_3 \cdot ... \cdot m_c}{|m_1 - 0||m_2 - 0||m_3 - 0|...|m_c - 0|} = 1$$
$$\pi_1(d) = \frac{m_1 \cdot m_2 \cdot m_3 \cdot ... \cdot m_c}{|0 - m_1||m_2 - m_1||m_3 - m_1|...|m_c - m_1|}$$
$$\pi_2(d) = \frac{m_1 \cdot m_2 \cdot m_3 \cdot ... \cdot m_c}{|0 - m_2||m_1 - m_2||m_3 - m_2|...|m_c - m_2|}$$

and so on, with $\pi_3(d), ..., \pi_n(d)$ following the same pattern. Now, consider the different input where we multiply each term by the same input $a \in \mathbb{Z}$.

$$\text{Then our input would become: d'} = \{0, am_1, am_2, am_3, ..., am_c\}. \tag{3.5}$$

From here we can again use our formula to calculate our output with this new degree sequence. Doing so would yield the following output:

$$\begin{aligned}
\pi_0(d') &= \frac{am_1 \cdot am_2 \cdot am_3 \cdot ... \cdot am_c}{|am_1 - 0||am_2 - 0||am_3 - 0|...|am_c - 0|} \\
&= \frac{a^c(m_1 \cdot m_2 \cdot m_3 \cdot ... \cdot m_c)}{a^c(m_1 \cdot m_2 \cdot m_3 \cdot ... \cdot m_c)} = 1 \\
\pi_1(d') &= \frac{a^c(m_1 \cdot m_2 \cdot m_3 \cdot ... \cdot m_c)}{|0 - am_1||am_2 - am_1||am_3 - am_1|...|am_c - am_1|} \\
&= \frac{a^c(m_1 \cdot m_2 \cdot m_3 \cdot ... \cdot m_c)}{a^c|0 - m_1||m_2 - m_1||m_3 - m_1|...|m_c - m_1|} \\
\pi_2(d') &= \frac{a^c(m_1 \cdot m_2 \cdot m_3 \cdot ... \cdot m_c)}{|0 - am_2||am_1 - am_2||am_3 - am_2|...|am_c - am_2|} \\
&= \frac{a^c(m_1 \cdot m_2 \cdot m_3 \cdot ... \cdot m_c)}{a^c|0 - m_2||m_1 - m_2||m_3 - m_2|...|m_c - m_2|},
\end{aligned}$$

again with $\pi_3(d'), ..., \pi_n(d')$ following the same pattern. From here it is straightforward to see that the a^c factor present in both the numerator and denominator of each output $\pi_i(d)$ will cancel out, leaving us with the same output as we got with the original degree sequence prior to multiplying by a. □

Example 3.7. Here we will prove Conjecture 2 for $c = 2$ by showing that if $n = \frac{d_2}{d_1}$ divides d_1 then all 3 numbers in the output sequence will be integers whose sum is greater than or equal to $2^c = 4$.

Proof. When $c = 2$ our input degree sequence will be of the form $d = \{0, d_1, d_2\}$. For this proof we will let $n = d_2 - d_1$, so our degree sequence will become $d = \{0, d_1, d_1 + n\}$. Next we note that when $c = 2$ our Π formula gives us the betti numbers $\beta_0 \cdot \{\pi_0, \pi_1, \pi_2\}$. Let $\beta_0 = \lambda$. For the first case, let's assume $n \mid d_1$. Then our betti numbers will be:

$$\pi_0 = 1$$

$$\pi_1 = \frac{d_1 + n}{n} = \frac{d_1}{n} + \frac{n}{n} = \frac{d_1}{n} + 1$$

$$\pi_2 = \frac{d_1}{n}.$$

Now since $n \mid d_1$ we know $\frac{d_1}{n}$ is an integer greater than or equal to 1. So $p_1 = \frac{d_1}{n} + 1$ must be greater than or equal to 2. By the same reasoning we know that $\pi_2 = \frac{d_1}{n}$ is an integer and at least 1. Then $\beta(M) = \pi_0 + \pi_1 + \pi_2 \geq 1 + 2 + 1 = 4$. So Conjecture 2 holds in this case. Next let's see what would happen if $n \nmid d_1$. In this case, our calculated π_1 and π_2 above will not be integers. Thus we will need to multiply them by an integer $\lambda \geq 2$. This will give us:

$$\pi_0 = \lambda$$

$$\pi_1 = \lambda(\frac{d_1}{n} + \frac{n}{n}) = \lambda(\frac{d_1}{n} + 1) = \frac{\lambda d_1}{n} + \lambda$$

$$\pi_2 = \lambda(\frac{d_1}{n}) = \frac{\lambda d_1}{n}.$$

Since $\lambda \geq 2$, we know π_0 is at least 2. Now $\pi_1 = \frac{\lambda d_1}{n} + \lambda$ must be at least 3 since $\frac{\lambda d_1}{n}$ must be an integer greater than or equal to 1 and λ is at least 2. Finally, by the logic in the previous sentence, $\pi_2 = \frac{\lambda d_1}{n}$ is at least 1. So the sum $\beta(M)$ in this case would be at least $2 + 3 + 1 = 6$, which is clearly greater than 4. Thus, in summary, we have shown that $\beta(M)$ will be:

$$\begin{cases} 4 & \frac{d_1}{n} = 1 \text{ (when } n \text{ divides } d_1) \\ \geq 6 & \frac{d_1}{n} \geq 2 \text{ (when } n \text{ does not divide } d_1) \end{cases}$$

This result actually proves Conjecture 3 as well, which states (for $c = 2$) that if $\beta(M) > 2^2 = 4$ then $\beta(M) \geq 2^2 + 2^1 = 6$. □

4 Macaulay 2 Software and pureBetti vs Betti Numbers

For the majority of the calculations and examples involving the normalized betti numbers $p_i(d)$ coming from the Π product formula, we rely on Macaulay2 to perform the calculations for us, rather than work through the formula by hand. The software is simple and user friendly. All we have to do after importing the BoijSoederberg package is type "pureBetti$\{0, d_1, d_2, ...d_n\}$" into the terminal, where $\{d_1, d_2, ...d_n\}$ is a list of positive integers in increasing order which represents the degree sequence for our module. Now, based on the name of the command, one might assume that the program will output the betti numbers of the module generated by the input values. As nice as it would be if this were true, it unfortunately is not the case. The 2008 Eisenbud-Schreyer Theorem however, reveals that the numbers output by Macaulay2 can still be of some use to us, by giving a precise statement about numbers produced from the Π product formula. In this section we will first see an example where our formula outputs fractions, which are not possible betti numbers. Then we will move on to examples where, although our output may be a sequence of integers, they violate the criteria of proven theorems and therefore are also not the actual betti numbers of any module.

Theorem 4.1. *Eisenbud-Schreyer Theorem[4] Given a list of numbers $\{\pi_0, \pi_1, ...\pi_i\}$ produced by the Π product formula, $N \cdot \{\pi_0, \pi_1, ...\pi_i\}$ will be the actual betti numbers of a module for some $N \in \mathbb{Z}$.*

Essentially, this theorem says that although $\{\pi_i(d)\}$ (the list of numbers output by the pureBetti command) may or may not be the betti numbers of a module, some integer multiple of the list is guaranteed to give the betti numbers. Let's look at a few examples of cases where the pureBetti output is and is not a list of actual betti numbers of a module, and how we can determine this.

Example 4.2. Here we will see a case where our formula yields a sequence of numbers which contains fractions. Say we choose the degree sequence $d = \{0, 1, 3\}$ as our input. If we use the Π product formula to calculate normalized pureBetti numbers by hand, We will get an output of $\{1, 3/2, 1/2\}$. Since betti numbers must be whole numbers, we know immediately that these are not the betti numbers of a module. Scaling by 2, however, gives us $\{2, 3, 1\}$, which are the betti numbers of a module. Macualay2 clears denominators for us automatically, which will be discussed in more detail and with proof later on. This example illustrates that if the numbers $\pi_i(d)$ are not integers we must first clear denominators to get a list of whole numbers before we can determine wheter or not we have a list of actual betti numbers. We shall see in the next example that unfortunately a list of whole numbers does not automatically mean these numbers are the betti numbers of a module.

Example 4.3. Now say we want to utilize the pureBetti function of Macaulay2 to calculate the betti numbers of the degree sequence $\{d_i\} = \{0, 1, 3, 4\}$. In this example c, the codimension of the object, is 3. The output given by the program will be $\{\pi_i(d)\} = \{1, 2, 2, 1\}$. Although these are all whole numbers, unlike those in the previous example, we will soon see some criteria which prove that they are not the betti numbers of any ideal. Applying the Eisenbud-Schreyer Theorem, however, we know that, for some integer N,

$$N\{1, 2, 2, 1\}$$

are the betti numbers of a module. In this case, $N = 2$ will work and $\{2, 4, 4, 2\}$ are the betti numbers of a module.

Example 4.4. You may be wondering how we can be sure whether a list of whole numbers are actually the betti numbers of an ideal or not. There are a few ways of knowing this, but we will first look at Mark Walker's 2018 result which shows that particular sums of betti numbers can only be achieved with specific values of β_i.

Theorem 4.5. *[5] (Walker 2018) Let M be a finitely generated module over a polynomial ring. If the characteristic of the ground field is not 2, meaning that $1 + 1 \neq 0$, then*

- *$\beta(M) \geq 2^c$ where $c =$ codim M*
- *If $\beta(M) = 2^c$, then $\beta_i = \binom{c}{i}$ for all i.*

Essentially, this result proves that Conjecture 2, which sets a lower bound for $\beta(M)$, is true for all values of c, and that in fact an even stronger constraint on this sum holds.

The other tools we can use to eliminate possible betti sequences can be summarized in the following theorem:

Theorem 4.6. *[1] Generalized Principal Ideal Theorem*

Suppose M is a module of codimension c. Then:

- $\beta_1(M) \geq \binom{c}{1}$;
- $\beta_c(M) \geq \binom{c}{c}$;
- $\beta_1(M) - \beta_0(M) + 1 \geq c$.

Example 4.7. Now that we have a set of tools for proving existence statements about modules, we can take another look at our earlier example where our calculated output was $\{\pi_i(d)\} = \{1, 2, 2, 1\}$. We claimed that these were not actually the betti numbers of any module, and applying the Generalized Principal Ideal Theorem shows us why this is true. In particular, the first formula in the Theorem, which states that $\beta_1(M) \geq \binom{c}{1}$, is violated by this sequence since $\pi_1(d) = 2$, which is not greater than or equal to $\binom{c}{1}$ which is 3. Thus our claim is verified and it was indeed necessary to multiply this output sequence by $N = 2$ to obtain an sequence of actual betti numbers.

Example 4.8. Now let's look at an example where $\beta(M) = 32$ and see how Walker's result can help us determine a lower bound for the integer, N, we must multiply our $\beta(M)$ by to obtain the actual betti numbers of a module. Suppose our degree sequence was $\{0, 2, 3, 7, 8, 10\}$. Then, up to an integer multiple, the betti numbers of any pure module with this resolution would be given by the Π product formula. The betti table below comes directly from Macaulay2 and shows that the calculated π_i sequence is $\{1, 7, 8, 8, 7, 1\}$.

$\beta(M)$	0	1	2	3	4	5
0	1	–	–	–	–	–
1	–	7	8	–	–	–
2	–	–	–	–	–	–
3	–	–	–	–	–	–
4	–	–	–	8	7	–
5	–	–	–	–	–	1

.

Now in this example $c = 5$ and it's easy to see that these output numbers violate the first of our big conjectures, which claims that $\{\beta_i\} \geq \{1, 5, 10, 10, 5, 1\}$.

So is this a counterexample disproving the conjecture, or are $\{1, 7, 8, 8, 7, 1\}$ simply not the betti numbers of an ideal? Note that for $\{\beta_i\} = \{1, 7, 8, 8, 7, 1\}$ and $\beta(M) = 32$. Walker's result, however, tells us that since $\beta(M) = 32 = 2^c$, it must be true that $\{\beta_i\} = \{\binom{c}{2}, \binom{c}{3}, ... \binom{c}{c-1}, \binom{c}{c}\} = \{1, 5, 10, 10, 5, 1\}$, which gives us a contradiction. Essentially then, the only way for $\beta(M) = 32$ is if $\{\beta_i\}$ is $\{1, 5, 10, 10, 5, 1\}$, so we know that $\{\beta_i\} = \{1, 7, 8, 8, 7, 1\}$ are not the betti numbers of any module. Thus, we know we must apply the Eisenbud-Schreyer Theorem and multiply this list of numbers by some integer N to obtain the betti numbers of a module. Walker's result implies that $N = 1$ doesn't satisfy our theorem, so we know that $N \geq 2$. Therefore the true betti numbers of this module must be at least $2 \cdot \{1, 7, 8, 8, 7, 1\} = \{2, 14, 16, 16, 14, 2\}$. Note that after we appropriately scale this set of numbers the resulting betti numbers will satsfy conjectures 1, 2, and 3, since $\beta(M)$ will become 64, which is greater than $2^c + 2^{c-1}$.

5 Concluding Thoughts

If you have read this far you now have a newfound understanding of a multitude of topics in Commutative Algebra, including varieties, ideals, and betti numbers. We have seen numerous theorems, examples, and proofs illustrating these topics. In addition, we have explored a few important open questions in the discipline. We analyzed specific modules and their sequences of betti numbers, which you now know how to calculate. Finally, we located potential counter-examples to 3 major conjectures and learned how to apply proven results to disprove their existence.

There is still much to explore on this topic, as infinitely many sequences would need to be checked in order to either prove our disprove the conjectures definitively. Research can also be done on modules with values of c which are greater than 5. Going forward, logical next steps in this area of study could focus on categorizing and finding commonalities among pseudo-counterexamples in order to formulate conjectures on them and perhaps find an actual counterexample.

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