Quiz 1 **Name:** ____________________

1. (4 points) Write does an expression using absolute value and inequalities that expresses: "The distance between x^2 and 9 is less than 4. Do not simplify.

2. (4 points) Draw on a number line the region represented by $|x+3| < 2$. Hint: first convert this to a sentence involving the word distance.

3. (4 points) Consider the statement

$$P_n\text{: "}(11)^n - 4^n \text{ is divisible by 7 when } n \text{ is a positive integer."}$$

If you wanted to prove this by mathematical induction, what would statement P_6 be?

4. (7 points) Consider the statement: "The sum of any finite number of even integers is even."

If you wanted to prove the above statement by induction, which would be a possible candidate for P_n? (Circle all that apply)

(a) The sum of any finite number of even integers is even

(b) The sum of n even integers is even.

(c) The sum of n integers is even.

(d) $1 + 2 + 3 + \ldots + n$ is even

(e) $2 + 4 + 6 + \ldots + 2n$ is even

(f) If $x_1, x_2, \ldots, x_n$ are even integers then so is $x_1 + \cdots + x_n$

(g) If we have a sum of finitely many even integers and add one more even integer then the result will still be even.

5. (5 points) Consider a collection of statements $P_1, P_2, P_3, \ldots$. Which of the following are true? (Circle all that apply)

   - Suppose we show that P_1 is true AND we proved that whenever P_n is true then so is P_{n+1}. Then it follows that P_{100} is true.
   - Let $N = 60$. Suppose we show that P_1 is true and that if P_N is true then P_{N+1} is true. Then P_n is true for all n.
   - Suppose we showed that whenever P_n is true then so is P_{n+1}. Then it follows that P_{100} is true.
   - Suppose that we showed that P_6 is true and that whenever P_n is true then so is P_{n+1} then it is true that P_N is true for all $N \geq 101$.
   - Suppose that we showed that P_2 is true and that whenever P_n is true then so is P_{n+1} then it follows that P_1 is true.

6. (2 points) In the following proof by induction, draw a star at the point in the proof when statement P_3 has been shown to be true.

Example 1
Prove $1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$ for positive integers n. □

Solution
Our nth proposition is

$$P_n\text{: “}1 + 2 + \cdots + n = \frac{1}{2}n(n+1)\text{.”}$$

Thus P_1 asserts $1 = \frac{1}{2} \cdot 1(1+1)$, P_2 asserts $1 + 2 = \frac{1}{2} \cdot 2(2+1)$, P_{37} asserts $1 + 2 + \cdots + 37 = \frac{1}{2} \cdot 37(37+1) = 703$, etc. In particular, P_1 is a true assertion which serves as our basis for induction.

For the induction step, suppose P_n is true. That is, we suppose

$$1 + 2 + \cdots + n = \tfrac{1}{2}n(n+1)$$

is true. Since we wish to prove P_{n+1} from this, we add $n+1$ to both sides to obtain

$$\begin{aligned} 1 + 2 + \cdots + n + (n+1) &= \tfrac{1}{2}n(n+1) + (n+1) \\ &= \tfrac{1}{2}[n(n+1) + 2(n+1)] = \tfrac{1}{2}(n+1)(n+2) \\ &= \tfrac{1}{2}(n+1)((n+1)+1). \end{aligned}$$

Thus P_{n+1} holds if P_n holds. By the principle of mathematical induction, we conclude P_n is true for all n. □

Quiz 2 **Name:** ____________________

1. (8 points) (Short Answer - no work is required)
   (a) What is the supremum of the set $(0,3)$?
   (b) Is your previous answer a max? (yes / no)
   (c) What is the infimum of the set $[4,7] \cup \{-3,1,9\}$?
   (d) Is your previous answer a min? (yes / no)
   (e) Which of the following are ordered fields? (Circle all that apply):

   $$\mathbb{Z} \qquad \mathbb{Q} \qquad \mathbb{R} \qquad \mathbb{C}$$

   (f) Which of the following are complete ordered fields? (Circle all that apply):

   $$\mathbb{Z} \qquad \mathbb{Q} \qquad \mathbb{R} \qquad \mathbb{C}$$

   (g) The completeness axiom is what allows us to say that every nonempty subset of $\mathbb{R}$ that is bounded above has a least upper bound. (True / False)
   (h) In the book there is a discussion about why it is true that every nonempty set that is bounded below has a greatest lower bound. This is true because
      i. that's what the completeness axiom says;
      ii. there's separate axiom that says this;
      iii. it can be **proven** using the completeness axiom
2. (4 points) A student writes the following statement which is not correct. Please explain why, by giving an explicit example (be as specific about, x, a, b) where the student's bound would not work.

   "Suppose that $x, a, b \in \mathbb{R}$ and $a < x < b$. Then I know that $|x| \leq |b|$."

3. (3 points) Suppose that $a \in \mathbb{R}$ and $\frac{1}{N} < a < M$ for some $N, M \in \mathbb{N}$. Then
   (a) The number a is [positive / zero / negative / we cannot tell]
   (b) Which of the following choices for k would make the statement $\frac{1}{k} < a < k$ necessarily true?

   Circle all that apply

      i. $k = \min(N, M)$
      ii. $k = \max(N, M)$
      iii. $k = N + M$
      iv. $k = NM$

4. (7 points) Sometimes / Always / Never. (Circle the best choice)

   (a) If M is an upper bound for S then $M+100$ is an upper bound for S [S / A / N]

   (b) If M is an upper bound for S then $2M$ is an upper bound for S. [S / A / N]

   (c) If M is an upper bound for S then $|M|$ is an upper bound for S. [S / A / N]

   (d) If $M<0$ is a lower bound for S then $2M$ is a lower bound for S. [S / A / N]

   (e) If $M<0$ is a lower bound for S then $|M|$ is an upper bound for S. [S / A / N]

   (f) If $M=\inf S$ and $M>0$ then $M/2$ is a lower bound for S [S / A / N]

   (g) If $M=\inf S$ then $M+1$ is a lower bound for S. [S / A / N]

5. (8 points) Let $M>0$ be a real number. Using the Archimedean Principle, prove that there is an $N\in\mathbb{N}$ such that

$$N^5-8N^3-N^2+4N>M.$$

   The proof outline is below.

Proof: Let $M>0$. By the AP ________________________________, then $N^5-8N^3-N^2+4N>$ $\square$

6. (5 points) Suppose that a,b,c are real numbers such that the distance between a and c is less than 10 and the distance between b and c is less than 3. Use the triangle inequality to prove that $|5a-c-4b|<62$. I will start the proof for you. I recommend you rewrite so that you can use the triangle inequality. Hint: (Try to group together stuff you know information about)

   Proof: Let x,y,z be as above. Then

$$|5a-c-4b|=$$

Quiz 3 **Name:** ______________________________

On this quiz, you may use the three term triangle inequality if you need it:

$$|\heartsuit + \Delta + P| \leq |\heartsuit| + |\Delta| + |P|.$$

1. Suppose that you know that $x, y \in \mathbb{R}$ and $|x-3| < 4$ and $|y-4| < 5$. Prove that

$$|3x - 10y| < 100.$$

Your proof should use the triangle inequality. Remember what you learned last week - try to incorporate $(x-3)$ and $(y-4)$ into your expression.

Proof: Let $x, y \in \mathbb{R}$ as given. Then

$$|3x - 10y| =$$

2. Prove that the sequence

$$s_n = \frac{4n}{n^2+2}$$

converges to 0.

The quiz continues on the back

3. Let $E = \{p \in \mathbb{Q} : p > 5\}$. Prove that $\inf E = 5$. Your proof should use the denseness of $\mathbb{Q}$.

Quiz 4 **Name:** ____________________

1. Writing examples:

   (a) (2 points) Write down a sequence a_n of positive numbers that converges to 0. (I want a formula).

   (b) (2 points) Write down a sequence b_n that converges to 7 such that all terms are > 7. (Hint: just add something to your sequence from part (a).

   (c) (2 points) Write down an example of a sequence c_n that does not converge.

   (d) (6 points) In the homework you proved that if $s_n \to 0$ and t_n is bounded then $s_n t_n$ is convergent and converges to 0.

   By writing down explicit examples for s_n and t_n, show that the same statement is NOT true if you only know that s_n converges (but not to 0) and that t_n is bounded. Write your answer in complete sentences. You don't need to prove anything, but explain why s_n t_n and $s_n t_n$ have the properties you want.

2. (8 points) Let a_n be a sequence and L a real number. Please answer Yes/No for the following (no explanation is needed)

   (a) Suppose that $\forall \epsilon > 0$ if $n > 1000/\epsilon$ then $|a_n - L| < \epsilon$. Does this imply that $\lim a_n = L$? [Yes/No]

   (b) Suppose that $\forall \epsilon > 0$ if $n \neq 4$ then $|a_n - L| < \epsilon$. Does this imply that $\lim a_n = L$? [Yes/No]

   (c) Suppose that

   $$\exists N \in \mathbb{N} \text{ such that } \forall \epsilon > 0, \text{ if } n \geq N, \text{ then } |a_n - L| < \epsilon.$$

   Does this mean that a_n converges to L?[Yes/No]

   (d) Suppose that $\lim a_n = L$. Then is it true that

   $$\exists N \in \mathbb{N} \text{ such that } \forall \epsilon > 0, \text{ if } n \geq N, \text{ then } |a_n - L| < \epsilon? \text{ [Yes/No]}$$

3. (10 points)
   Let a_n, b_n be sequences. Suppose that $a_n \to a$ and $b_n \to b$. Define the sequence $c_n = 3a_n - 4b_n$.

   Prove (using the ϵ, N definition) that $c_n \to 3a - 4b$.

   Proof:

Quiz 5 **Name:** ______________________________

1. (10 points) Turn in your Writeup about the Torero Mystery Hallowe'en Netflix Theorem.

2. **Definitions** (6 points)

   (a) Let S be a subset of $\mathbb{R}$. Define what it means for L to be the supremum of S.

   (b) Define what it means for a sequence t_n to converge.

3. **What can you use?** (6 points) Suppose you know that $\lim a_n = -5$. If you are writing a proof (of some other statement) then there are lots of ways that you can USE this fact. Which of the following can you use? (I.e. which of these must be true). Mark your selections, no justification is needed:

   a.) "a_n is bounded."

   b.) "There exists an $M \in \mathbb{R}$ so that $|a_n| \leq M$ for all n.

   c.) "There exists an $M \in \mathbb{R}$ with $M > 0$ so that $|a_n| \leq M$ for all n.

   d.) "Let $\epsilon > 0$. There exists an N so that if $n > N$ then $|a_n + 5| < \epsilon/100$"

   e.) "There exists an N so that if $n \leq N$ then $|a_n + 5| < 0.1$"

   f.) "There exists an N so that if $n \geq N$ then $|a_n + 5| < 0.1$"

   g.) "There exists an N so that if $n > N$ then $|a_n + 5| < 0.1$"

   h.) "Let $\epsilon > 0$. There exists an N so that if $n > N$ then $|a_n + 5| < \epsilon/n$"

   i.) "Let $\epsilon > 0$. Then there exists an n such that $|a_n+5| < \epsilon/100$"

   j.) -5 is the minimum of $\{a_n\}$

   k.) -5 is the infimum of $\{a_n\}$

   l.) "Let $\epsilon > 0$. Then for all n, $|a_n + 5| < \epsilon$"

4. (3 points) Circle all that apply to the sequence $a_n = \frac{(-1)^n}{n}$

   - a_n is increasing
   - a_n is decreasing
   - a_n is monotone
   - a_n is bounded
   - a_n is convergent

5. **Giving Examples** (3 points each)

   (a) True or False: There exists an increasing sequence s_n such that $s_n \in \mathbb{Z}$ and $s_n < 0$ for all n. If there is, given an example, if there is not, briefly explain why.

(b) Write down two sequences a_n and b_n such that $\lim a_n = 0$ and $\lim b_n = \infty$ but $\lim(a_n \cdot b_n) = 6$. Or explain why this is not possible.

(c) Write down a **convergent** sequence a_n satisfying $|a_n| < n \cdot (0.01)$ for all $n \in \mathbb{N}$.

(d) Write down a **divergent** sequence a_n satisfying $|a_n| < n \cdot (0.01)$ for all $n \in \mathbb{N}$.

(e) Write down a sequence such that $\lim a_n \neq \infty$ but such that a_n is not bounded above. (You can use "…" in your example if the pattern is clear.)

6. (10 points) Suppose that a_n is a sequence that converges to 7. Prove that there is some N such that for $n > N$, $a_n > 3$. I am looking for an argument that uses the **definition** of the limit, with ϵ. A number line may help.

Quiz 6 **Name:** ______________________________

1. (4 points each) Give examples of the following. If no such example exists, **explain why, citing any theorems**

   (a) Two divergent sequences a_n and b_n such that $a_n + b_n$ converges

   (b) A sequence that is monotone but not Cauchy

   (c) A sequence s_n of numbers between 4 and 7 such that s_n has NO convergent subsequence

2. (8 points) For each of the following, mark True or False. (No explanation needed)

   (a) If $a \le x_n \le b$ for all n then $|x_n| \le b$ for all n. [T / F]

   (b) If a sequence converges then it is Cauchy. [T / F]

   (c) If a sequence is Cauchy then it converges. [T / F]

   (d) All convergent sequences are bounded. [T / F]

   (e) All bounded sequences converge [T / F]

   (f) If x_n converges to L then **every** subsequence of x_n converges to L. [T / F]

   (g) Suppose that x_n is a Cauchy sequence. Then the sequence $b_n = 2x_n - x_{n+3}$ is Cauchy. [T / F]

   (h) If $|a_n|$ converges then a_n converges. [T / F]

3. (8 points) Let $\epsilon = 1$, and let $N \in \mathbb{N}$ be arbitrary. Prove there are two numbers $n, m > N$ such that $|\sqrt{n} - \sqrt{m}| > \epsilon$. (Hint: this can be very short!)

3. Let S be a subset of $\mathbb{R}$. It is possible that S obeys the following property:

$$(\star) \qquad \forall x \in S \ \exists r > 0 \ : \text{ the interval } (x - r, x + r) \text{ is a subset of } S.$$

(a) (8 points) Explain why the closed interval $[2, 8]$ does NOT have this property. I'm just looking for a few short sentences with a clear explanation. You must be specific in your reasoning.

(b) (8 points) Let $S = (2, 8)$ be the open interval. Prove that the set S satisfies property $(\star)$ Take a deep breath - you've got this - you've been practicing all semester to solve these sorts of problems. Hint: Your proof can be quite short, you might only need a sentence or two.) I'll get you started:

Proof: Let $x \in S$ be an arbitrary element of S.

Quiz 7 - 20 points total **Name:** ______________________________

1. (7 points) Prove, using sequences that the following function $f : \mathbb{R} \to \mathbb{R}$ is not continuous. Write your answer in complete sentences.

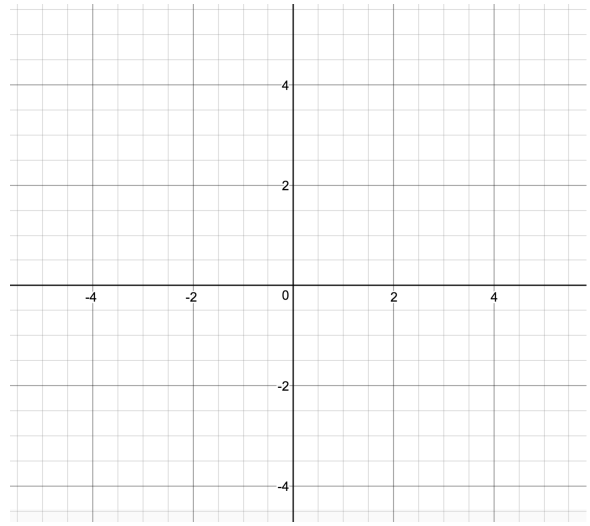
$$f(x) = \begin{cases} x^2 - 4 & \text{if } x > -1 \\ x + 1 & \text{if } x \leq -1 \end{cases}$$

2. (6 points) Prove, **using sequences (yes an explicit formula for x_n)** that the function $f : \mathbb{R} \to \mathbb{R}$ is NOT continuous at $x = 4$.

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ x^2 & \text{if } x \text{ is irrational.} \end{cases}$$

3. (5 points) Let $D = (-\infty, -2] \cup \{0\} \cup [2, \infty)$ and define

$$f : D \to \mathbb{R}, \quad f(x) = \begin{cases} 4 & \text{if } x = 0 \text{ or } x = 2 \\ 2 & \text{otherwise} \end{cases}$$



Draw a graph of $f(x)$. Please use open/closed circles where appropriate.

Is f continuous at 0? (Yes / No)

Is f continuous at 2? (Yes / No)

Is f continuous at -2? (Yes / No)

4. (2 points) A student is studying the function defined below:

$$f : (-3, 3) \to \mathbb{R}, \quad f(x) = \begin{cases} 1 - x & \text{if } x \le 2 \\ 2x & \text{if } x > 2 \end{cases}$$

And wants to prove that f is NOT continuous at $x = 2$. The student writes: "I will show that f is not continuous at $x = 2$. Note that $f(2) = -1$. I will find a sequence of inputs x_n that converge to 2 such that $f(x_n)$ does NOT converge to -1. Here is my sequence:

$$x_n = 2 + 10/n.$$

Then $f(x_n) = 2(2 + 10/n) = 4 + 20/n$ which converges to 4. Since this is not equal to -1, we are done.

There is a tiny error in this proof, but it is major enough that the proof is not valid. What is the error, and how could you fix it? (Hint: there's a simple fix).

Quiz 8 (20 points total) **Name:** ______________________________

1. Short Answer

   (a) (1 point) Suppose that $x > y$, then which of the following is a correct simplification of $|x - y|$?

   $x - y$ $\quad$ $y - x$ $\quad$ we can't tell

   (b) (1 point) Suppose that $x > y$, then which of the following is a correct simplification of $|3-(x+y)|$?

   $3 - x - y$ $\quad$ $x + y - 3$ $\quad$ we can't tell

   (c) (2 point) Suppose you are writing an ϵ, δ proof and you know that $3 < x < 4$. What does this mean about $\frac{1}{x^2}$? I want you to write down the best (correct) inequality you can for what $\frac{1}{x^2}$ could be.

   (d) (2 points) Let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) = 7$. Let $\epsilon = .0001$. Which of the following values of δ would "work" to show continuity at a point x_0 (meaning that IF $|x-x_0| < \delta$ THEN $|f(x)-f(x_0)| < \epsilon$.) (Circle all that apply.)

3	$\epsilon/3$	$\epsilon/7$	.00000000001
10^{2023}	9	$-\epsilon/3$	it depends on x_0.

2. (8 points - Quadratic Proof) Prove, using δ and ϵ that the function $f(x) = 2x^2 - x$ is continuous at $x = -2$. Make sure you clearly say what δ is.

3. (5 points) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function satisfying the following condition for all x:

$$|f(x)| \leq |x|.$$

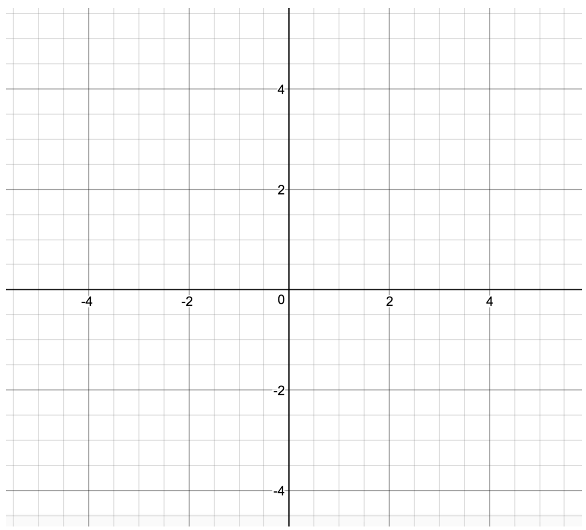
A) What does this mean $f(0)$ must be? Why?

B) Prove that f must be continuous at $x = 0$. (Hint: just trust the process and use your scaffolding of your ϵ, δ proofs! You can do it!)

Quiz 9 **Name:** ______________________

1. (5 points) Write down an explicit function (meaning give an formula for $f(x)$) AND draw a graph such that f has the following properties. Hint: Draw the graph first and don't make it too complicated.

   - f has domain $[-2, 4]$
   - f is bounded
   - f does NOT achieve a maximum value.
   - f achieves a minimum when $x = 3$.



$$f(x) = \begin{cases} \\ \end{cases}$$

2. (6 points) The following are all slightly different statements - for each of them, determine whether or not the statement is true and write True or False. No explanation is necessary.

   (a) If $f : [a, b] \to \mathbb{R}$ is continuous then $\forall y$ between a and b there exists $c \in [a, b]$ such that $f(c) = y$.

   (b) If $f : [a, b] \to \mathbb{R}$ is continuous then $\forall y$ between $f(a)$ and $f(b)$ there exists $c \in [a, b]$ such that $f(c) = y$.

   (c) If $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) < f(b)$ then there exists $c \in [a, b]$ such that $f(a) < f(c) < f(b)$.

   (d) If $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) < f(b)$ then there exists $c \in [a, b]$ such that $f(c) = f(b) - 0.0001$.

   (e) If $f : [a, b] \to \mathbb{R}$ is continuous then there exists $c \in [a, b]$ such that $f(c) = \frac{1}{2}(f(a) + f(b))$.

   (f) If $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) < f(b)$ then there exists $c \in [a, b]$ such that $f(b) > f(c) > f(b) - 0.0001$.

   (g) Prove, using δ and ϵ that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \frac{1}{x^2+1}$ is continuous at $x = 1$.

3. (7 points) Suppose that f is continuous on $\mathbb{R}$ and you know $f(0) > 0$ and $f(2) < 3$. Show that there is some number c such that $f(c) = c^2$.

4. (7 points) Prove, using ϵ, δ that the function $f(x) = \frac{2x}{x-2}$ is continuous at $a = 3$. Hint: To avoid making mistakes, start by finding $f(a)$ and double check it. In your proof you should find $|x-3|$ appearing at some point to use your δ. The "other" factor should be in the denominator, but you practiced how to deal with that on Tuesday's HW. You can do it!

Quiz 10 (20 points) **Name:** ______________________________

Big Theorems about Uniform Continuity

1. A **continuous** function on a **closed** bounded interval is **always** uniformly continuous on that interval.
2. A Lipschitz function is uniformly continuous on its domain.
3. A continuous function $f(x)$ is uniformly continuous on a bounded open interval if and only if it can be extended to the include the end points so that the extension is continuous. (Think about "filling in values for $f(a)$ and $f(b)$ that make the whole function continuous on the closed interval).
4. A uniformly continuous function on any bounded interval (a, b) or $[a, b]$ must be bounded.
5. If f is differentiable (has a derivative) on its domain and this derivative is **bounded**, then f is uniformly continuous. This works even on infinite intervals.

For each of the following write down an example of a function (include domain!) that has the given property. If no such function exists, say why. You only need to explain your answer if it says so.

1. A function that is NOT continuous.

2. A function that is continuous but not uniformly continuous on its domain. Explain why it is not uniformly continuous.

3. An continuous unbounded function that is uniformly continuous.

4. A continuous function with bounded slope that is not uniformly continuous.

5. A continuous function that is not Lipschitz but that is uniformly continuous.

6. A continuous function with unbounded slope that is not uniformly continuous.

7. A continuous bounded function that is not uniformly continuous. Explain briefly why it is not not.

8. A continuous function f and a sequence of inputs x_n (in the domain) such that $f(x_n)$ does not converge.

9. Let $\delta > 0$. Write down two real numbers x and y such that $|x - y| < \delta$. (Obviously x and y can depend on δ.)

10. Let $\delta > 0$. Find a real number x so that the numbers x and $y = x + \delta/2$ satisfy

$$|x^2 - y^2| \geq 1.$$

Show any necessary work.

Quiz 11 (20 points) **Name:** ________________________________

1. (2 points) State Rolle's Theorem

   (2 points) State Fermat's Theorem

2. (6 points) Consider the following functions defined from $\mathbb{R}$ to $\mathbb{R}$.

$$f(x)=\begin{cases}\sin(1/x) & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases},\quad g(x)=\begin{cases}x\sin(1/x) & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases},\quad h(x)=\begin{cases}x^2\sin(1/x) & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases},$$

$$j(x)=|x+2|,\qquad k(x)=x^2-4x.$$

In the table below, mark a box if the function has the indicated property.

	is continuous at 0	is differentiable at 0	is continuous at 2	is differentiable at 2
$f(x)$				
$g(x)$				
$h(x)$				
$j(x)$				
$k(x)$				
$h'(x)$				

3. Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = 3x^2 + 2x$. Suppose that $a, b \in [0, 3]$.

   (a) (7 points) Using the Mean Value Theorem, prove that $|f(a) - f(b)| \leq 100|a - b|$. Make sure you explain what hypotheses you are using.

   (b) (3 points) Is f a contraction mapping? Why or why not? (Be careful and specific in your explanation)

Quiz 12 **Name:** ____________________

This quiz is due in class on Wednesday December 6th.

Please write your solutions carefully and explaining why any hypotheses apply. I will be more strict than with timed in-class quizzes.

You may use your notes, our book / readings / previous homework assignments. You may also brainstorm and work together, but please write up your own solutions.

There are **two** problems each worth ten points - there's a front and a back.

1. Let $f : [0, 3] \to \mathbb{R}$ be given by $f(x) = -x^3 + 6x^2 + x$. By using the Mean Value Theorem, prove that if $a, b \in [0, 3]$ then
$$|f(b) - f(a)| \leq 15|b - a|.$$

2. Suppose that t_n is a sequence of real numbers and $t_n \to 0$. Let

$$y_n = t_n(\sin n).$$

Prove, using any method you like, that y_n converges to 0. You may use that $-1 \le \sin n \le 1$.

Definition Quiz: We say that a real number g is an upper bound for a set P if

We say that a real number b is a lower bound for a set T if

We say that a set A is bounded above if

We say that a set B is bounded below if

We say that a set S is bounded if

A real number g is called the supremum of C (or the least upper bound of C) if

A real number h is called the infimum of S (or the greatest lower bound of S) if

Let S be a set then $\sup S = \infty$ means

Let S be a set then $\inf S = -\infty$ means

The completeness axiom says

State the theorem that says: "The Rational Numbers are Dense in $\mathbb{R}$".

The sequence s_n converges to L if

The sequence s_n converges if

When we write $\lim a_n = \infty$ this means

When we write $\lim a_n = -\infty$ this means

We say s_n is increasing if

We say s_n is decreasing if

We say s_n is monotone if

The Monotone Convergence Theorem says that

The sequence s_n is called Cauchy if

The Bolzano Weierstrass Theorem says

(Sequence Definition) A function $f : D \to \mathbb{R}$ is continuous at b if

(ϵ, δ Definition) A function $f : D \to \mathbb{R}$ is continuous at b if

A function $f : D \to \mathbb{R}$ is continuous if

The Intermediate Value Theorem says

A function $f : S \to \mathbb{R}$ is called uniformly continuous on S if

A function $f : (a, b)$ is called differentiable at c if $c \in (a, b)$ and

The Extreme Value Theorem says that

Fermat's Theorem says that

Rolle's Theorem says that

The Mean Value Theorem says that

1 Real Analysis Review Packet

In this class we have focused on the following topics:

- Foundations of the Real Numbers

  1. (Thank U Next) How does induction work? Can you write a short induction proof? Can you analyze an induction proof and say how it is working? Can you answer short answer questions about induction?
  2. What are the axioms of the real number system, what is a field? an ordered field? can you give examples of each?
  3. What does the completeness axiom say? How is it related to the Archimedean Principle? How can we use these?
  4. What's sup? inf? How do these compare to max/min. Does every set have a sup? Every bounded set? Does every set have a max?
  5. What does the triangle inequality say? Can you use it? What about if you need to add/subtract stuff first?
  6. For this topic and others make sure you can give lots of examples.

  > This section covers packets Day 1 - Day 7 and Quizzes 1- 3.

- Sequences

  1. What is the definition of convergence? Can you give examples of sequences that have certain properties?
  2. Can you prove basic properties about sequences? If a sequence is convergent, why is it bounded? If a_n and b_n converge why does $a_n + b_n$ converge? What about $a_n b_n$? If $a_n b_n$ converges, does a_n? Same question with $a_n b$ where b is a constant? Given lots of similar looking statements, which of them are equivalent to the definition of convergence? Which are stronger? Which are weaker? There was a quiz question about this. Why does $(-1)^n$ not converge?
  3. Can you write a proof using the ϵ, N definition of convergence.
  4. Can you use the main limit theorems in a proof? The squeeze theorem? These might show up in a proof on continuity using sequences, e.g. the ones at the end of Day 8.
  5. What does it mean to say $\lim a_n = \infty$? $-\infty$? Is this the same as saying that a_n is unbounded? (no) why not? Can you give examples of a sequence with one property but not the other?
  6. What does the Monotone Convergence Theorem say? How do we use it? What about those recursive sequence proofs where we showed something converged and we needed to find the limit? (Day 14).
  7. When we used the monotone convergence theorem we used it to prove that e exists. How did that go? How did we conclude that $2 < e < 3$? At some point you proved that the geometric series $1 + r + r^2 + \cdots = 1/(1-r)$ if $-1 < r < 1$. How did that proof go? If you encounter a geometric series in the wild (i.e. on the final) can you use this formula correctly? Can you use this to prove that e is irrational? Can you compare something to a geometric series?
  8. Then we defined Cauchy sequences. Can you prove that a certain sequence is Cauchy? Is Cauchy equivalent to convergent? Yes. Which direction was easier to prove? (One was short, the other

took a few days. You should be able to prove the shorter one) Why are Cauchy sequences bounded? Does the harmonic series $1 + 1/2 + 1/3 + \cdots$ converge? Does the sequence of partial sums of the harmonic series converge? is it Cauchy?

9. I won't ask you nasty questions about subsequences and the technicalities about the subscripts, but I will ask you to give examples, state the Bolzano-Weierstrass theorem, and answer questions about it. You should also be prepared to use BW if you need it in a proof.

This section covers packets Day 8 - Day 18 and Quizzes 4-6.

- Continuity and Uniform Continuity

  1. What are the two equivalent ways to define what it means to be "continuous at a". What does it mean for a function to be "continuous". Can you write proofs using sequences that a function is/is not continuous? Can you write a proof using ϵ/δ?
  2. What does the intermediate value theorem say? Can you solve problems like on the HW and quizzes?
  3. What was uniform continuity all about? What does it mean for closed intervals, for open intervals? What's the connection with slope. Quiz 10 will be helpful to review.

  This section covers packets Day 19-30 and Quizzes 7-10.

- Limits of Functions and Derivatives

  1. Can you state (using, N, ϵ, δ etc what it means for say $\lim_{x\to 3^+} f(x) = -\infty$? What about $\lim_{x\to -\infty} f(x) = 2$?
  2. Can you use the definition of the derivative to calculate derivatives, say of things like $x^2 + 3x$ or of extensions of $x^2 \sin(1/x)$?
  3. Can you state and use Fermat's Theorem, Rolle's Theorem, the Mean Value Theorem like on the homework and quizzes?

  This section covers packets Day 31-34 and Quiz 11 - 12

- Integration

  1. What does it means for $f : [a, b] \to \mathbb{R}$ to be integrable? Can you answer conceptual questions about $U(f), L(f), U(f, P), L(f, P)$?
  2. Can you give an example of a function that is not integrable and sketch a reason why?
  3. Do you know what the fundamental theorem of calculus says (the three parts?) pay attention to the hypotheses and conclusions.
  4. Can you explain why a simple function is integrable by finding a partition P so that $U(f, P) - L(f, P) < \epsilon$?

  This section covers packets Day 35-41

A bunch of problems to work on. And a goose.


IVT

1. **PLEASE READ FIRST:** Before looking at any of these supplemental problems, please go through all of the quizzes that you have completed. These are the BEST way to prepare. For instance, on quizzes we have had 2 questions about using the MVT in a proof. This means that's a good problem to study. I might not have a similar problem below, but that's because it's already on the quizzes twice!

   The problems below are lots of extra random practice. Some of them were HW, some were problems I put on old midterms/finals and others were ones I have collected from over the years. I'll probably include some of these on the final. Please do not feel the need to solve any (definitely not all) of these, as there are so many. But you may want to flip through and see what you find interesting. The difficulty ranges from easy to difficult. I will not provide solutions to these, but am happy to go over any of these during office hours.

2. Find the inf / sup / max / min (if they exist) for the sets:
$$S = \{1 - 2/n : n \in \mathbb{N}\}$$
.
$$T = S \cap [-1/2, 1/2]$$
$$U = \{x \in \mathbb{Q} : x^2 < 2\}.$$

3. Write down an infinite sum that converges. Explain why you know.

4. Suppose that $b \in \mathbb{N}$ prove that the sum below converges by comparing it with a geometric series. Be precise in your algebra
$$S = \frac{1}{b+3} + \frac{1}{(b+3)(b+4)} + \frac{1}{(b+3)(b+4)(b+5)} + \cdots$$
   Can you show that S is not an integer?

5. Use the ϵ, δ definition of continuity to prove that $x^2 - 3x$ is continuous at $x = 1$.

6. Use the ϵ, δ definition of continuity to prove that $\frac{1}{x^2}$ is continuous at $x = 2$.

7. Using δ, ϵ prove that $f : [0, \infty) \to \mathbb{R}$ given by $f(x) = \sqrt{x}$ is continuous.

8. Use the ϵ, δ definition of continuity to prove that $\frac{1}{x^2+1}$ is continuous at $x = 1$.

9. Write down an example of a sequence x_n that is unbounded but such that $\lim x_n$ is neither ∞ or $-\infty$. Explain in your own words why not.

10. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function, not necessarily continuous everywhere. Suppose that f is continuous at 2. Show that there is some $\delta > 0$ such that f is bounded on the interval $[2-\delta, 2+\delta]$. Write your proof clearly.

11. Suppose that b_n is a sequence converging to zero. Suppose that a_n is a sequence such that for any $n, m \in \mathbb{N}$, if $m \geq n$ then $|a_m - a_n| \leq |b_n|$. Prove that a_n is Cauchy.

12. Suppose that a_n is a sequence that converges to L. Prove that the sequence defined below converges to L also. Hint: get a common denominator and share the love!

$$b_n = \frac{a_1 + a_2 + \cdots a_n}{n}$$

13. Let $f : [a, b] \to \mathbb{R}$ be a continuous function with the property that for each $x \in [a, b]$ there is a $y \in [a, b]$ such that $|f(y)| \leq \frac{1}{2}|f(x)|$. Prove that there is a sequence x_n in $[a, b]$ such that $f(x_n) \to 0$. Then use this to prove that there is a point $c \in [a, b]$ such that $f(c) = 0$.

14. Can you find a function f and a sequence x_n as in the previous example so that x_n does not converge? Why or why not?

15. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function whose range is contained in $\mathbb{Q}$. That means its outputs are ALL rational numbers. Prove that f must be a constant function. If you want a hint, look at the label of the goose.

16. Prove that the function

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 1 - x & \text{if } x \notin \mathbb{Q} \end{cases}$$

is continuous at $x = 1$ and discontinuous elsewhere.

17. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous. Suppose that for every $\alpha > 0$ there is an M such that if $|x| > M$, then $|f(x)| < \alpha$. Prove that f is uniformly continuous.

18. Prove that the sum of two rational numbers is a rational number.

19. Prove that the infimum of the set $\{1/n\}$ is 0. Use the AP in your proof.

20. Prove that the sequence $(-1)^n(1 - 1/n)$ diverges.

21. Prove that the sequence $(-1)^n(3/n)$ converges using ϵ, N.

22. Prove that if $x_n \to L$ and c is a constant then $cx_n \to cL$.

23. Suppose that x_n is a sequence of integers which converges to L. Prove that L is an integer and that there is an N such that for all $n > N$, $x_n = L$.

24. Suppose that $x_n \to 9$ prove that there is an N such that for all $n > N$, $x_n > 4$.

25. Suppose that x_n is an increasing sequence and suppose x_{n_k} is a subsequence that converges to L. Prove that $x_n \to L$.

26. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and that $f(q) = 0$ for all $q \in \mathbb{Q}$. Prove that $f(x)$ is zero for ALL real numbers x.

27. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and suppose that $f(x_n) \to f(c)$. Does it follow that $x_n \to c$? Give a proof or a counterexample.

28. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $x_n \to b$ for some b, does $f(x_n)$ necessarily converge? Why or why not?

29. If $f : [1, 3] \to \mathbb{R}$ is continuous and $x_n \to b$ for some b, does $f(x_n)$ necessarily converge? Why or why not?

30. If $f : (1, 3) \to \mathbb{R}$ is continuous and $x_n \to b$ for some b, does $f(x_n)$ necessarily converge? Why or why not?

31. Can you give an example of a function that has no maximum on $[2, 3]$? A continuous one? Why or why not?

32. Can you give an example of a function that is not continuous?

33. Can you give an example of a Lipschitz function with Lipschitz constant $K = 0.4$?

34. Prove, using the definition of Cauchy that $n/4^n$ is Cauchy.

35. Let a_n be a sequence and suppose that for each n, $|a_n| \le n$. Show that $r_n = a_n/n$ has a convergent subsequence.

36. Let $a_n \to a$ and $b_n \to b$ be convergent sequences. Using ϵ, N, prove that $a_n - 5b_n$ converges to $a - 5b$.

37. How did we define e in class? Prove that e is not an integer.

38. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function with the property that there is a sequence x_n with $1 \le x_n \le 10$ such that $f(x_n) > n$. Explain why f cannot be continuous.

39. Given an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ with the property that there is a sequence x_n such that $f(x_n) > n$.

40. Let $n \ge 0$ be an integer. Define the function $f(x) = x^n \sin(1/x)$ if $x \ne 0$ and $f(0) = 0$. For which n is f continuous? differentiable? When is the derivative of f continuous? Can you use the limit definition to find the derivative.

41. Suppose that $f : (-1, 1) \to \mathbb{R}$ is a continuous function but not necessarily differentiable. Suppose that $f(0) = 1$. Define $g(x) = xf(x)$. Determine whether g is differentiable at zero. You may use that a function $A(x)$ is continuous at c if and only if $\lim_{x \to c} A(x) = A(c)$.

42. Suppose that $f : [0, \infty) \to \mathbb{R}$ is continuous and that it is differentiable on $(0, \infty)$. Suppose that $f(0) = 0$ and $M > 0$ is such that $|f'(x)| < M$ for all $x > 0$. Show that for all $x \ge 0$,
$$|f(x)| \le M|x|.$$

43. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function that satisfies
$$|f(x) - f(y)| \le |x - y|^2$$
for all $x, y \in \mathbb{R}$. Suppose that $f(3) = 5$. Find, with proof, what $f(7)$ must be. Hint: Try to calculate $f'(a)$.

44. Suppose that f is uniformly continuous on $(0, 1)$. Does $f(1/n)$ converge? Why or why not? If you only know that f is continuous, but not uniformly continuous, does your answer change?

45. True or False? With explanation/counterexample
Every sequence has an increasing subsequence?
Every sequence has a bounded subsequence
every unbounded sequence has a subsequence with limit $\pm\infty$.
If a sequnce has a greatest term (meaning a max) then every subsequence has a greatest term (meaning a max).

46. Suppose that $x_n \to 0$ and y_n is bounded. Prove that $x_n y_n \to 0$. Note you can't use the main limit theorems here. Make sure you understand why.

47. Using the definition of Cauchy, prove that if x_n and y_n are Cauchy sequences then so is $x_n + y_n$ and $x_n y_n$.

48. Same question with convergent.

49. Is the same true if you replace convergent with divergent? Why or why not?

50. Prove that $\sqrt{n}$ is NOT Cauchy.

51. Let $a \in \mathbb{R}$ and define

$$f(x) = \begin{cases} a & \text{if } x = 0 \\ \frac{1}{x} & \text{if } x \neq 0 \end{cases}$$

Prove that $f(x)$ is not continuous at 0 and that $f(x)$ is continuous for all $x \neq 0$.

52. Prove that $\sqrt{n}$ does satisfy the property that for all ϵ there is an N such that for all $n > N$, $|a_{n+1} - a_n| < \epsilon$. Does this condition imply that the sequence is Cauchy? Does $sqrtn$ converge?