1. Week 1 Problems

(1) Suppose that you have 6 identical toothpicks that are all the same length. Can you arrange them so that they form 4 identical triangles?

(2) A) What is the largest (finite) number of intersection points that a line and a hexagon can have? Draw a picture showing this. How do you know your number is maximum? (A hexagon is a polygon with six straight line segments of sides.)

B) Same as part A) but with a heptagon (7 sides) instead of a hexagon? How can you justify your answer?

(3) At a conference attended by a huge number of people, there are a lot of handshakes, and everyone might shake hands with different numbers of people. Let's say that n people have shaken hands an even number of times, and m people have shaken hands an odd number of times. Explain why the number m has to be even.

(4) A) You are given 80 coins that are all identical except for 1 counterfeit coin that is lighter than the others. Locate the counterfeit coin by using four weighings on a balance. (A balance will compare two things and tell you which is heavier.)

B) (Warning: This is tough!) Among 12 similar coins there is one counterfeit. It is not known whether the counterfeit coin is lighter or heavier than a genuine one. Using three weighings on a pan balance, how can the counterfeit coin be identified? (In this problem you don't need to figure out if it is heavier or lighter, just find the counterfeit one)

C) (Even Tougher - but you can do it!) In the same setup as problem B) still in only 3 weighings, can you determine the counterfeit coin AND whether it is heavier or lighter?

(5) Find (with explanation) all real number solutions to $x^2 + \sqrt{x} = 18$. Hint: I bet you can find at least one solution! Hint2: See the footnote on the second page.

(6) A traveler, having no money, but owning a gold chain having 7 links is accepted at an inn on the condition that they pay 1 link each day of his stay. If the traveler is to pay daily, but may take change in the form of previously paid links, and if they remain seven days, what is the least number of links that must be cut out of the chain? (Note: A link may be taken from any part of the chain.)

(7) Refer to the following number triangle

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&1&1&&& \\ &&1&2&3&2&1&& \\ &1&3&6&7&6&3&1& \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{array}$$

Each number is the sum of the three numbers (above left, above, above right) in the previous row. Prove that in every row, beginning with the third row, there must be an even number somewhere.

(8) Five sailors and a monkey were shipwrecked on an island. They spent the first day gathering coconuts. During the night, one sailor woke up and decided to take his share of the coconuts. He divided them into five piles. One coconut was left over so he gave it to the monkey, then hid his share and went back to sleep.

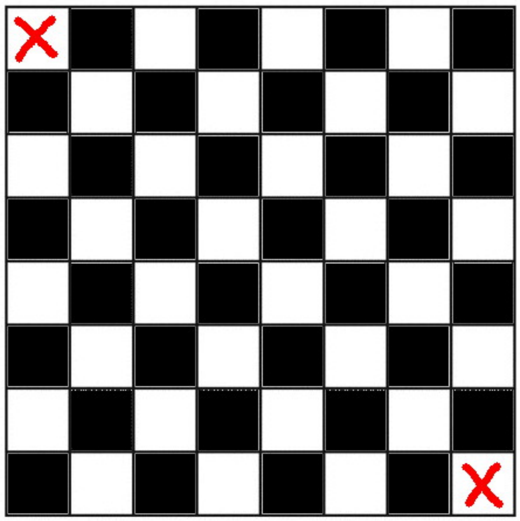
Soon a second sailor woke up and she did the same thing. After dividing the coconuts into five piles, one coconut was left over which she gave to the monkey. She then hid her share and went back to bed. The third, fourth, and fifth sailers followed exactly the same procedure. The next morning, after they all woke up, they divided the remaining coconuts into five equal shares. This time also 1 coconut was left. What is the smallest number of coconuts that could have been there in the original pile? *

*Let $y = \sqrt{x}$ and try factoring, given what you know about hint 1.

2. Week 2 Problems

(9) I have 10 sticks in my bag. The length of each stick is an integer. No matter which three sticks I try to use, I cannot make a triangle out of those sticks. What is the minimum length of the longest stick?

(10) Consider the chessboard below that is missing the two corner squares:



Can you cover the chess board using 2×1 dominoes? Why or why not?

(11) A) Imagine you have 25 pebbles, each occupying one square (and one square only!) on a 5 by 5 chess board. Suppose that each pebble must move to an adjacent square by only moving up, down, left, or right. (And the outcome still have exactly one pebble on each square). Is this possible? Explain your reasoning.

B) What if you were allowed to also move a pebble diagonally?

C) What happens in parts A and B when instead of a 5 by 5 chessboard it is an n by n chessboard? What happens for different values of n?

(12) A prisoner who was sentenced to death is given a last chance to survive. He receives 2 identical boxes and 5 balls, 2 red and 3 black. He is to split the balls between the two boxes as he wants. Then he randomly chooses one of the two boxes and randomly picks a ball from the chosen box. If he picks a red ball he survives. How should he split the balls to maximize the probability of survival?

(13) The function f is defined on the set of integers and satisfies

$$f(n) = \begin{cases} n-3 & \text{if } n \geq 1000 \\ f(f(n+5)) & \text{if } n < 1000 \end{cases}$$

Determine $f(84)$.

(14) Mary told John her score on a problem-solving exam, which was over 80. From this information, John was able to determine the number of problems that Mary answered correctly. If Mary's score were any lower, but still over 80, John could not have determined this. What was Mary's score?

Scoring system for the exam: The exam had 30 problems and the total score was determined by the formula

$$s = 30 + 4c - w$$

where c is the number of correct answers and w is the number of incorrect answers. The subtraction is to penalize for guessing, and there is no problem with leaving problems blank. Note that some of the problems can be left blank, which is not the same as incorrect. So for example, if Mary answered 20 correct, and missed 2 and left 8 blank, her score would be $30 + 4(20) - 2 = 108$.

3. Week 3 Problems

(15) 100 different points are given in the plane. Does there always exist a line so that each of the two half planes determined by the line contain 50 points?

(16) A mouse eats its way through a $3 \times 3 \times 3$ cube of cheese by tunneling through all of the 27 $1 \times 1 \times 1$ subcubes. If the mouse starts in one corner and always move to an uneaten subcube, can it finish at the center of the cube?

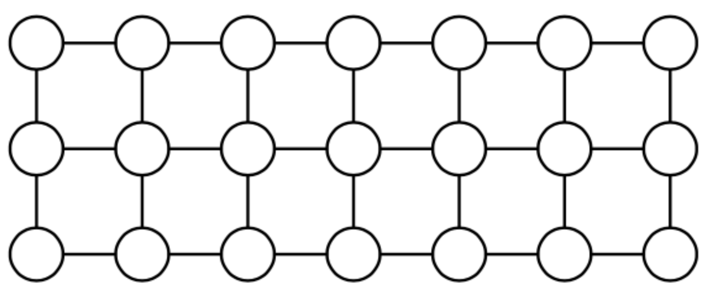
(17) Imagine a hallway with 1000 doors numbered consecutively 1 through 1000. Suppose all of the doors are closed to start with. Then some dude with nothing better to do walks down the hallway and opens all of the doors. Because the dude is still bored, he decides to close every other door starting with door number 2. Then he walks down the hall and changes (i.e., if open, he closes it; if closed, he opens it) every third door starting with door 3. Then he walks down the hall and changes every fourth door starting with door 4. He continues this way, making a total of 1000 passes down the hallway, so that on the 1000th pass, he changes door 1000. At the end of this process, which doors are open and which doors are closed?

(18) 10 prisoners are to be lined up all facing the same direction. On the back of each prisoners head, the jailer places either a black or a red dot. Each prisoner can only see the color of the dot for all of the prisoners in front of them and the prisoners do not know how many of each color there are. The jailer may use all black dots, or perhaps he uses 3 red and 7 black, but the prisoners do not know. The jailer tells the prisoners that if a prisoner can guess the color of the dot on the back of their head, they will be set free, but if they guess incorrectly, they will be remain in prison. The jailer will call on them in order starting at the back of the line. Before lining up the prisoners and placing the dots, the jailer allows the prisoners 5 minutes to strategize. **What plan can the prisoners devise that will maximize the number of prisoners that survive?** Each prisoner can only respond with the word black or red.

Follow-up questions:

What if there are n prisoners? What if there are more than 2 colors?

(19) In the lattice below, we color 11 vertices points black. Prove that no matter which 11 are colored black, we always have a rectangle with black vertices (and vertical and horizontal sides).



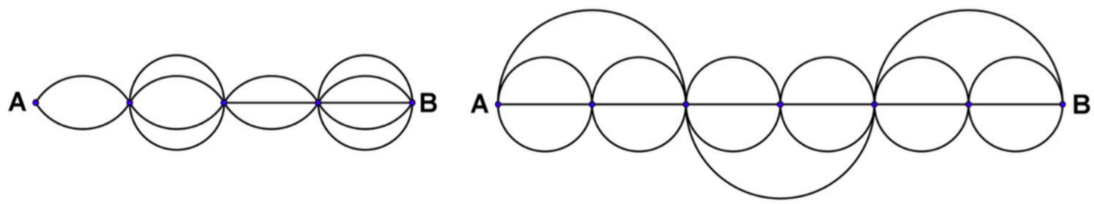
(20) Each point of the plane is colored red or blue. Show that there is a rectangle whose vertices are all the same color.

4. Week 4 Problems

(21) Find a visual proof of the following fact. For $n \in \mathbb{N}$:

$$1 + 3 + 5 + \ldots + (2n - 1) = n^2$$

(22) How many different paths are there on these two maps from A to B if we may only move from left to right?



(23) How many ways can 110 be written as the sum of 14 different positive integers? Hint: First figure out what the largest possible integer in the sum could be.

(24) Here is a magic trick: think of at most 7 prime numbers which are greater than 10. (Some of your numbers can be the same). Calculate the sum of the squares of your primes and I will tell you how many primes you picked! How does this trick work?

(25) You and a partner are playing a game where you take turns placing pennies on a circular table. The pennies have to lay flat and cannot overlap. Once the pennies are placed, they cannot be moved. If you cannot place a penny on the table on your turn you lose. What is your strategy to win? Do you want to go first or second? Would the answer change if there was a rectangular table?

(26) Infinitely many points are given on the plane. Does there always exist a line so that each of the two half-planes defined by the line contains infinitely many points? (this builds on problem 17)

(27) You have 14 coins, dated 1901 through 1914. Seven of these coins are real and weigh 1.000 ounce each. The other seven are counterfeit and weigh 0.999 ounces each. You do not know which coins are real or counterfeit. You also cannot tell which coins are real by look or feel. Fortunately for you, Zoltar the Fortune-Weighing Robot is capable of making very precise measurements. You may place any number of coins in each of Zoltar's two hands and Zoltar will do the following:

- If the weights in each hand are equal, Zoltar tells you so and returns all of the coins.
- If the weight in one hand is heavier than the weight in the other, then Zoltar takes one coin, at random, from the heavier hand as tribute. Then Zoltar tells you which hand was heavier, and returns the remaining coins to you.

5. Week 5 Problems

This week we'll start with a short lesson on the **Pigeonhole Principle**. Please check out our shared folder (on blackboard) for a nice PDF of all sorts of problems and applications of the this principle.

(28) Show that among any 11 positive integers, there must be at least two such that their difference is divisible by 10.

(29) 51 integers are chosen between 1 and 100 inclusively. Prove that 2 of the chosen integers are consecutive.

(30) 10 integers are chose from 1 to 100 inclusively. Prove that we can find 2 disjoint non-empty subsets of the chosen integers such that the 2 subsets give the same sum of elements.

(31) There are some people (more than one person) at a party. Prove that 2 of them have the same number of friends at the party. (It is assumed that friendship is mutual).

(32) There are 5 points in a square of side length 2. Prove that there exist 2 of them whose distance it at most $\sqrt{2}$.

(33) Determine all possible values of $A^3 + B^3 + C^3 - 3ABC$

(34) Find c if a, b, c are positive integers which satisfy $c = (a + bi)^2 - 107i$ where $i^2 = -1$.

(35) What is the largest even integer which cannot be written as the sum of two odd composite numbers? (Recall that a number is called composite if it is divisible by at least one positive integer other than itself and 1.)

6. Week 6 Problems

(36) Which equations below have solutions with $x, y \in \mathbf{Z}$?

$$x^2 = 4y + 0$$
$$x^2 = 4y + 1$$
$$x^2 = 4y + 2$$
$$x^2 = 4y + 3$$

(37) Find all positive integer solutions to $x^2 = 3y + 2$.

(38) Find all positive integer solutions to

$$\frac{1}{a} + \frac{1}{b} = \frac{3}{2018}.$$

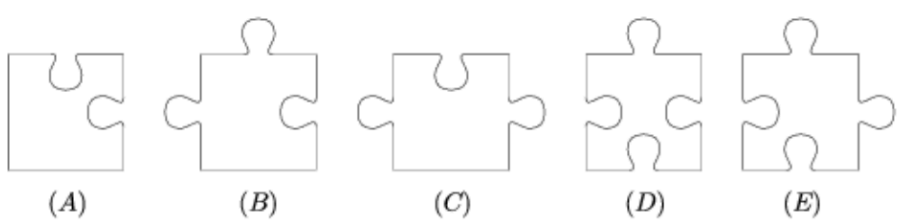
The prime factorization of 2018 is $2 \cdot 1009$.

(39) Let a, b, c, d, e, f, g, h be distinct elements in the set $\{-7, -5, -3, -2, 2, 4, 6, 13\}$. What is the minimum possible value of $(a+b+c+d)^2 + (e+f+g+h)^2$?

(40) Consider the equation below. If a is a number, what number is it?

$$a = \cfrac{2}{2 + \cfrac{2}{2 + \cfrac{2}{2 + \cfrac{2}{2 + \cfrac{2}{2 + \cdots}}}}}$$

(41) A rectangular puzzle that says "850 pieces" actually consists of 851 pieces. Each piece is identical to one of the 5 sample pieces shown below. How many pieces of type (E) are there in the puzzle?



(42) Find odd positive integers $A < B < C$ such that

$$\frac{1}{3} = \frac{1}{A} + \frac{1}{B} + \frac{1}{C}.$$

7. Week 7 Problems

(43) (a) Put 10 dots on a 4×4 array so that each row and each column contains an even number of dots.

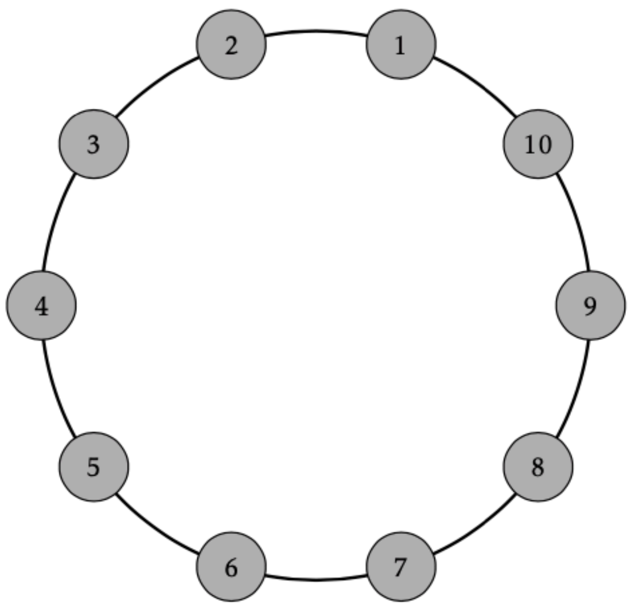
(b) Put dots on a 4×5 array so that each row and each column contains an odd number of dots.

(43) (Repeat, but a good one to work on. Hint in footnote†) Find odd positive integers $A < B < C$ such that

$$\frac{1}{3} = \frac{1}{A} + \frac{1}{B} + \frac{1}{C}.$$

(44) In a PE class, everyone has 5 friends. Friendships are mutual. Two students in the class are appointed captains. The captains take turns selecting members for their teams, until everyone is selected. Prove that at the end of the selection process there are the same number of friendships within each team.

(45) Ten people form a circle. Each picks a number and tells it to the two neighbors adjacent to him in the circle. Then each person computes and announces the average of the numbers of his two neighbors. The figure shows the average announced by each person. What is number picked by the person who announced 6?



(46) (Putnam A1) Find the smallest positive integer j such that for **every** polynomial $p(x)$ with integer coefficients and for every real number k, the integer

$$p^{(j)}(k), \text{that is, the } j\text{th derivative of } p(x), \text{ evaluated at } k$$

is divisible by 2016.

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†What is the largest and smallest possible value for A? What about B, C?

8. Week 8 Problems

Additional Homework this week:
I'd like everyone to try writing up a solution to one of the problems from this week very carefully. Part of our final project will be writing up solutions to some of our favorite problems as a class, and I'd like to give you practice writing a careful solution and give you feedback next week.

- Write your answer on a separate sheet of paper from your other homework.
- Use complete sentences that show your solution.
- The goal is to communicate the "why" of the problem.
- I'd love to chat in office hours or email if you have any questions.
- Imagine you were writing up a solution that would appear in a math puzzle magazine.
- If you would rather write up a solution to a problem from a previous week, feel free!
- Remember - one of the goals in this course is to push our limits and challenge ourselves, so I encourage you to consider committing to solving a more challenging problem and try writing it up. You can do it!

(47) (Related to Problem 44) Is it possible in a 5×5 array of integers that

  (a) the row sums are greater than 300 but the column sums are less than 200.

  (b) the row products are greater than 300 but the column products are less than 200.

  (Remember that integers can be positive or negative.)

(48) Is it possible to label the edges of a cube with the numbers $1, 2, \ldots, 12$ so that

  (a) every vertex sum is equal?

  (b) every face sum is equal?

  (The vertex sum is the sum of the three numbers on the edges starting from the vertex. The face sum is the sum of the four numbers on the dges bounding the face.)

(49) Annie, Bob, and Cristy are sitting by a campfire when Cristy announces that she is thinking of a 3-digit number. She then tells Annie and Bob that the number she is thinking of is one of the following:

$$515, 516, 519, 617, 618, 714, 716, 814, 815, 817.$$

  Next, Cristy whispers the leftmost digit in Annie's ear and then whispers the remaining two digits in Bob's ear. The following conversation then takes place:

  Annie: I dont know what the number is, but I know Bob doesnt know too.

  Bob: At first I didnt know what the number was, but now I know.

  Annie: Ah, then I know the number, too.

  What was Cristy's number?

(50) (Putnam Training about Polynomials) The following are taken from: `https://tinyurl.com/putnampoly` (which is a collection of problems that build up to Putnam problems, using warmups, problems from High School Olympiad competitions etc. They are all pretty challenging, but this PDF comes with hints and solutions. Feel free to consult these resources. Sometimes it's a good idea to read through some problems and solutions. Sometimes understanding a solution is hard work.

1) Find a polynomial with integral coefficients such that $\sqrt{2} + \sqrt{5}$ is a zero.

2) Let $p(x)$ be a polynomial **whose coefficients are all integers**. Further, suppose that there are distinct integers a, b, c such that $p(a) = p(b) = p(c) = -1$.

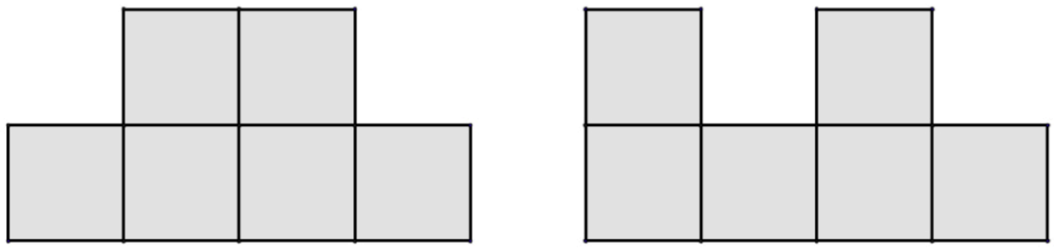
A) Can you find any zeros of $f(x) = p(x) + 1$?

B) If so, what can you conclude about the factors of $f(x)$?

C) Use this to conclude that $p(x)$ cannot have an integral root.

9. Week 9 Problems

(51) In the figure below, we can see the front view and the side view of a building built from unit cubes. What is the minimal and maximal number of cubes in such a building?



(52) How many numbers can you select from the first 100 positive integers if the product of the selected numbers is not allowed to be divisible by 72?

(53) The people on a planet in a far away galaxy worships whole numbers. (Whole numbers are the same throughout the universe!) They used to have three magic numbers: 1, 7, 16. Later everybody wanted to have their own magic number and so they introduced two laws to generate new magic numbers:

(a) if x and y are magic then xy is also magic

(b) if x and y are magic then $2x - y$ is also magic (x and y can be the same number).

Is 123456 or 1234567 magic?

(54) On the top floor of a castle lives a mathematician. The floor has 17 bedrooms arranged in a row. Each bedroom has doors connecting to the adjoining bedrooms as well as to the outside corridor. The mathematician sleeps in a different bedroom each night by opening the door to an adjoining bedroom and spending the night and the next day in that room.

One day a USD Math Major arrives at the castle and wants to ask a question to the mathematician. The guardian angel at the castle tells the math major of the mathematician's sleeping patterns and informs that each morning she may knock on one of the outside doors. If the mathematician happens to be behind that door, then the student will get to ask the question - success! Can the student guarantee meeting the mathematician? and if so, what is a possible strategy?

10. Week 10 Problems

(55) You are presented with a magical basket of eggs, all of which have the property that they will not break if they are dropped from the Nth floor of a 100-floor building (or from any lower floor). Your job is to figure out what that floor N is. (It's the same for all the eggs). The building manager will let you into the building to perform a test for \$1 per drop.

a) What would your strategy be if you were given only one egg to perform your tests (and if it breaks... well I hope you know what N is by the point, or else you're out of eggs!) What is the maximum number of test drops you might need to perform? (In other words, in the worst case, how much will you have to pay the building manager?)

b) What would your strategy be if you had two eggs? (If one breaks, you still have another egg!) What is the maximum number of test drops you might need to perform the drops? (In other words, in the worst case, how much will you have to pay the building manager?)

(56) What is the maximum number of pieces that you can cut a circular pancake into with n straight cuts?

(57) On a circle, draw n points on the boundary and connect these points with straight lines. For each n, what is the largest possible number of regions you can create?

11. Week 11 Problems

(58) How many numbers can you select from the first 100 positive integers if no two of the selected numbers are coprime (relatively prime)?

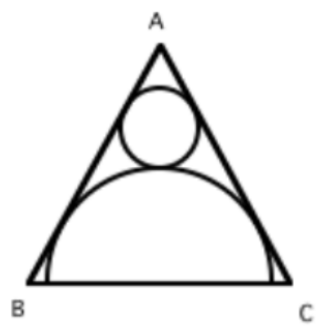
(59) You and your two friends Thor and Valkyrie are captured by Loki. In order to gain your freedom, Loki sets you the following challenge. The three of you are put in adjacent cells. In each cell is a quantity of stones. Each of you can count the number of stones in your own cell, but not in anyone elses. You are told that each cell has at least one stone but at most nine stones, and no two cells have the same number of stones. The rules of the challenge are as follows: The three of you will ask Loki a single question each, which he will answer truthfully Yes or No. Every one hears the questions and the answers. Loki will set all of you free only if one of you can correctly determine the total number of stones in all the cells. Here is the initial conversation.

Thor: Is the total an even number? Loki: No.

Valkyrie: Is the total a prime number? Loki: No

You have five stones in your cell. What question will you ask? You should assume that Thor and Valkyrie are just as good at logic as you are.

(60) The figure below shows an equilateral triangle ABC with an inscribed semicircle of radius R that is tangent to sides AB and AC, and inscribed circle of radius r that is tangent to the triangle and the semicircle. Find the value of r/R.



(61) There are five students at a party. We ask how many friends they have in the group. Here are the answers:

Alex: I have 4 friends.

Bob: I have fewer friends than Alex has.

Camille: I have as many friends as Doug.

Doug: Eric has one more friend than I have.

Eric: I have an odd number of friends.

Question: Are Camille and Doug friends?

12. Week 12 Problems

(62) Suppose that n is a positive integral and $2n + 1$ and $3n + 1$ are both perfect squares. Show that n is a multiple of 40.

(63) There is an island with 100 inhabitants, 50 with blue eyes and 50 with brown eyes. They can all see the color of other peoples eyes, but not their own. They cannot communicate about it and there are no mirrors, and there is a law that says that if someone discovers that they have blue eyes, then that person must leave the island at midnight that night. All the inhabitants of the island are very good at logical reasoning. One day, an outsider arrives on the island, takes one look at its inhabitants and says: I am pleased to have seen at least one person with blue eyes. What happens?

(64) 100 prisoners are sentenced to life in prison in solitary confinement. Upon arrival at the prison, the warden proposes a deal.

The warden has a large bowl containing the names of all the prisoners. Each day he randomly chooses one prisoner from the bowl, the corresponding prisoner is taken to the interrogation room, and the cell number is returned to the bowl.

While in the interrogation room, the prisoner will not be allowed to touch anything except the light switch, which the prisoner may choose to turn on or off.

The prisoner may make the assertion that all 100 prisoners have been in the room. If the prisoners assertion is correct, all prisoners will be released. If the prisoner is incorrect, the game is over and their chance to be freed is gone.

The prisoners are given one meeting to discuss a strategy before their communication is completely severed. What strategy should they adopt in order to ensure, with 100

The initial state of the light is OFF when the first prisoner enters the room.

Follow-up questions:

What if the prisoners didn't know whether the light was going to be on or off at the start?

What if the warden waited a random amount of time before calling the first prisoner?

Can you find a solution that still works in these cases?

(65) In a parlor game, the magician asks one of the participants to think of a three digit number (abc) where a, b, and c represent digits in base 10 in the order indicated. The magician then asks this person to form the numbers (acb), (bca), (bac), (cab), and (cba), to add these five numbers, and to reveal their sum, N. If told the value of N, the magician can identify the original number, (abc). Play the role of the magician and determine the (abc) if $N = 3194$.

(66) The polynomial

$$1 - x + x^2 - x^3 + \cdots + x^{16} - x^{17}$$

may be written in the form

$$a_0 + a_1 y + a_2 y^2 + \cdots + a_{16} y^{16} + a_{17} y^{17},$$

where $y = x + 1$ and the a_i's are constants. Find the value of a_2.