ABSTRACT

The Betti numbers of a graded module over a commutative ring are an important invariant that provide information on the graded free resolution of the module. In this thesis we will study a seminal theorem by Jürgen Herzog and Michael Kühl which establishes a correspondence between the degree sequence of a finite graded free resolution of dimension zero and its Betti numbers. Along the way, we will provide a basic introduction to these objects and how they harmonize in another theorem due to David Hilbert and Jean-Pierre Serre concerning the dimension of a module over a polynomial ring and an algorithm for its computation.

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1. INTRODUCTION

A notoriously difficult problem in mathematics is establishing a rigorous definition for dimension. People tend to have a reasonable geometric and physical intuition for this problem, especially up to three dimensions. We may think of this in terms of the degrees of freedom an object has or in how many different directions could an ant walk on an object. We know that a point is 0-dimensional because it has no freedoms at all. An ant standing on a point wouldn't be able to move in any direction without exiting that point. A line or line segment has 1 dimension because it has length and an ant would be able to move along it in one direction. Likewise, a plane is 2-dimensional and rocks, humans, and all of space are 3-dimensional. Unfortunately, this intuition has its limitations. Consider the points in space that could be reached by a robotic arm with two joints, a "shoulder" and an "elbow." What is the dimension of the points in space this arm can reach? If neither joint is operational, it can only reach one point. If only one joint is operational, if will be able to reach points on a curve, likely a circle of the operational joint moves radially. Furthermore, if both of its joints are operational, it could either reach points on a disk if its shoulder joint can only move radially or any point in space within the radius of of its arm length it its shoulder is a socket like ours. In a situation like this, our solution can either be zero, one, two, or three-dimensional depending on the conditions so it's not as easy to define. In algebraic geometry we study questions like this by examining the solution sets of systems of polynomial equations and their geometric properties such as dimension. Let's see a more concrete example.

Example 1.1. Suppose we have the system of equations

$$\begin{aligned} xy &= 0 \\ yz &= 0 \\ zx &= 0. \end{aligned}$$

One can check that the solution to this system is just the union of the x, y, and z axes, which are all 1-dimensional objects. Intuition would dictate that the dimension of this solution set should be 1. A canonical strategy used by algebraic geometers is to decompose the solution set into a nested chain of its irreducible components. In our case, this works as we would hope because we get the chain

$$\{\text{origin}\} \subset \{\text{line through the origin}\}$$

of length 1 so this definition agrees with our intuition that the solution set of this system would be 1-dimensional.

Alternatively, we can generate some combinatorial data on these polynomials. We have three equations $\{xy, yz, zx\}$ all of degree 2. We want to know if there are polynomial-valued vectors $(c_1, c_2, c_3)^T$ such that $c_1(xy) + c_2(yz) + c_3(zx) = 0$. We find that there are two such linearly independent vectors, namely

$$\mathbf{v}_1 = \begin{pmatrix} z \\ -x \\ 0 \end{pmatrix} \text{ and } \mathbf{v}_2 = \begin{pmatrix} 0 \\ x \\ -y \end{pmatrix}$$

and these vectors bring our polynomials up to degree 3. We could then ask if there are any more vectors that cancel these two vectors but in this case there are not. We can compile this information into the following table.

$$\{xy, xz, yz\} \Longrightarrow \begin{array}{|ccc} \\ \hline 1 & - & - \\ - & 3 & 2 \end{array}.$$

We can encode this data into a rational function

$$Q(t) = \frac{1-3t^2+2t^3}{(1-t)^3} = \frac{(1-t)^2(1+2t)}{(1-t)^3} = \frac{1+2t}{(1-t)^1}.$$

David Hilbert and Jean-Pierre Serre proved that the degree of the denominator of these functions gives a definition of dimension that is equivalent to the geometric one given above. A big advantage of this technique is that it gives us an algorithmic procedure for calculating dimension that can be programmed into a computer.

Let's take a look at another example in more variables.

Example 1.2. Suppose we have the system

$$\begin{aligned} vw &= 0, \quad wx = 0 \\ xy &= 0, \quad yz = 0 \\ zv &= 0. \end{aligned}$$

This time, the solution to our system is the union of five 2-dimensional planes meeting at the origin. Using the traditional geometric strategy, we get the chain

$$\{\text{origin}\} \subset \{\text{line}\} \subset \{\text{plane}\}$$

of length 2 so our dimension is 2. We have 5 polynomials of degree 2 and it turns out there are 5 vectors $\mathbf{v}_i$ such that

$$(vw, wx, xy, yz, zv) \cdot \mathbf{v}_i = 0$$

where this product has degree 3. In this case, there actually is a vector that will cancel these 5 vectors and bring them up to degree 5. Just as before, we construct a table

$$\{vw, wx, xy, yz, zv\} \Longrightarrow \begin{array}{|cccc} & & & \\ \hline 1 & - & - & - \\ - & 5 & 5 & - \\ - & - & - & 1 \end{array}$$

which leads to the function

$$Q(t) = \frac{1 - 5t^2 + 5t^3 - t^5}{(1-t)^5} = \frac{(1-t)^3(t^2+3t+1)}{(1-t)^5} = \frac{t^2+3t+1}{(1-t)^2}.$$

The degree of the denominator is 2 so this system is 2-dimensional which agrees with what we found using the geometric approach.

The previous examples have done well to illustrate the equivalence of the these two definitions of dimension. Let's move on to a case where the geometric definition is less viable.

Example 1.3. Suppose we have

$$\begin{aligned} a^2 &= 0, & b^2 &= 0 \\ ac + bd &= 0, & ae + bf &= 0. \end{aligned}$$

It is not obvious how the solution set to this system looks so our geometric strategy will be difficult to implement but our computational method remains unfazed. With the help of a computer we get the table

$$\{a^2, b^2, ac+bd, ae+bf\} \Longrightarrow \begin{array}{c|ccccccc} & & & & & & & \\ \hline & 1 & - & - & - & - & - & - \\ & - & 4 & - & - & - & - & - \\ & - & - & 13 & 20 & 15 & 6 & 1 \end{array}$$

and

$$\begin{aligned} Q(t) &= \frac{1-4t^2+13t^4-20t^5+15t^6-6t^5+t^6}{(1-t)^6} \\ &= \frac{(1-t)^2(1+2t-t^2-4t^3+6t^4-4t^5+t^6)}{(1-t)^6} \\ &= \frac{1+2t-t^2-4t^3+6t^4-4t^5+t^6}{(1-t)^4} \end{aligned}$$

so this system is 4 dimensional.

We see that not only is this definition of dimension equivalent to the geometric one but it enables us to efficiently compute dimension regardless of how difficult it is to visualize the solution set of a system of polynomials With this strategy in our arsenal we can venture further to uncover more information on sets of polynomials like those above. We would like to understand the relations among these polynomials and how cancellations occur among them. In doing so, we will create something to the effect of a polynomial family tree. We would also like to understand how this information is encoded in tables like the ones in the previous examples and what information is given by the numbers within them. What are the possible shapes of these tables? Is there always only one number in each column and if not, what does that mean? What are the bounds, if any, of the values within these tables? Can we develop a complete classification of the Betti numbers? Questions like these are the focus of an area of study known as Boij-Söderberg theory and at the heart of them lies a seminal theorem by Jürgen Herzog and Michael Kühl called the *Herzog-Kühl equations* which were given as Theorem 1 in [1]. In this paper, we will explore these equations, the mathematical machinery involved in their development, and motivate how these equations apply to Conjecture 2.4 in [2] by Mats Boij and Jonas Söderburg which spurred the development of Boij-Söderberg theory and were proven by David Eisenbud and Frank-Olaf Schreyer as Thoerems 0.1 and 0.2 in [3].

2. COMPLEXES AND RESOLUTIONS

A primary goal of algebra is to develop concrete and rigorous methods for describing mathematical objects and their properties. This sometimes requires more care than one might expect. One particularly powerful way of approaching this is by studying the ways in which one object can be mapped to another. In this section, we will dig deep into that notion and explore what we can learn about objects by studying the properties of the maps among them.

2.2 COMPLEXES

Let R be a commutative ring such as $\mathbb{C}[x,y,z]$.

Definition 2.1. A *complex* $\mathbf{C}$ is a sequence of homomorphisms of R-modules

$$\mathbf{C} : \cdots \longrightarrow C_{i+1} \xrightarrow{\delta_{i+1}} C_i \xrightarrow{\delta_i} C_{i-1} \xrightarrow{\delta i-1} \cdots$$

such that $\delta_{i-1}\delta_i = 0$ for all i. We call the set $\delta = \{\delta_i\}$ the *differential* of $\mathbf{C}$. The (i-th) *homology* of $\mathbf{C}$ is $H_i(\mathbf{C}) = \ker(\delta_i)/\operatorname{im}(\delta_{i+1})$ and the complex is said to be *exact* at i if $H_i(\mathbf{C}) = 0$ or, equivalently, if $\ker(\delta_i) = \operatorname{im}(\delta_{i+1})$.

The definition above is fairly dense so let's begin to process it with some simple examples.

Example 2.2. Let $R = \mathbb{Z}$ and $C_i = \mathbb{Z}$ for all i. Let $\delta_i = 0$ for i even and δ_i is multiplication by a prime number p for i odd. We have

$$\cdots \longrightarrow \mathbb{Z} \xrightarrow{(0)} \mathbb{Z} \xrightarrow{(p)} \mathbb{Z} \xrightarrow{(0)} \mathbb{Z} \xrightarrow{(p)} \mathbb{Z} \longrightarrow \cdots$$

Clearly, since $\mathbb{Z}$ is a $\mathbb{Z}$-module, multiplication by 0 and p are homomorphisms, $p(0) = 0$ and $0(p) = 0$, this is a complex. Furthermore, since $\operatorname{im}(\delta_{\text{even}}) = 0$, $\operatorname{im}(\delta_{\text{odd}}) = p\mathbb{Z}$, $\ker(\delta_{\text{even}}) = \mathbb{Z}$, and $\ker(\delta_{\text{odd}}) = 0$,

$$H_{\text{even}} \cong \mathbb{Z}/p\mathbb{Z} = \mathbb{Z}_p \text{ and } H_{\text{odd}} \cong 0/0 = 0.$$

Example 2.3. Let $R = \mathbb{Z}$ and $C_i = \mathbb{Z}_{p^2}$, the integers modulo p^2 for some prime number p. If δ_i is multiplication by p for all i then we have

$$\cdots \longrightarrow \mathbb{Z}_{p^2} \xrightarrow{(p)} \mathbb{Z}_{p^2} \xrightarrow{(p)} \mathbb{Z}_{p^2} \xrightarrow{(p)} \mathbb{Z}_{p^2} \longrightarrow \cdots .$$

Multiplication by p is a homomorphism and $\delta_{i-1}\delta_i = pp = p^2 \equiv 0 \pmod{p^2}$ so this is a complex. Since $\ker(\delta_i) = p\mathbb{Z}_{p^2} \cong \mathbb{Z}_p = \mathrm{im}(\delta_i)$, $H_i \cong \mathbb{Z}_p/\mathbb{Z}_p = 0$ for all i. This is an example of an infinite exact complex.

Note that in order for a sequence $0 \longrightarrow A \xrightarrow{f} B$ to be exact, since the image of the zero map is always zero, the kernel of f must be zero so

$$0 \longrightarrow A \xrightarrow{f} B \text{ is exact } \iff f \text{ is injective.}$$

Likewise, for a sequence $B \xrightarrow{g} C \longrightarrow 0$ to be exact, since the kernel of the zero map is all of C, the image of g must be all of C so

$$B \xrightarrow{g} C \longrightarrow 0 \text{ is exact } \iff g \text{ is surjective .}$$

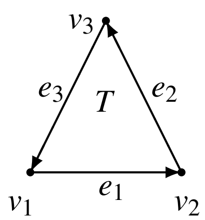
In this way, we get a *short exact sequence*, which is of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

so f is injective, g is surjective and $\mathrm{im} f = \ker g$. We will use this idea in a proof later on in the paper.

We see in the above examples that the conditions for a sequence of maps to be a complex are fairly weak. Complexes are the tip of a vast iceberg that extends from commutative algebra to topology to graph theory and much more. Before we go on, let's explore how complexes show up in an interesting intersection of these areas and see what happens when our maps aren't quite so trivial.

Example 2.4. Suppose T is a triangle with oriented edges as below.



This triangle induces a complex given by assigning $C_0 = \mathbb{Z}^3$ with basis $\{v_1, v_2, v_3\}$,

$C_1 = \mathbb{Z}^3$ with basis $\{e_1, e_2, e_3\}$, and $C_2 = \mathbb{Z}$ with basis $\{T\}$. We define maps via

$$0 \longrightarrow \mathbb{Z} \xrightarrow[T \mapsto e_1+e_2+e_3]{} \mathbb{Z}^3 \xrightarrow[\substack{e_1 \mapsto v_2 - v_1 \\ e_2 \mapsto v_3 - v_2 \\ e_3 \mapsto v_1 - v_3}]{} \mathbb{Z}^3 \longrightarrow 0 .$$

We can think of this complex as mapping each component (face, vertices, and edges) to their boundary components. For example, the face maps to the sum of the edges at its boundary and the edges map to the difference of the vertices at their boundaries with signs chosen so the resulting maps form a complex.

We see that complexes naturally show up in many interesting areas. We will show that when our complexes are exact everywhere, they become a powerful mathematical microscope that will allow us to learn quite a lot about the objects we put under it.

2.2 FREE RESOLUTIONS

From this point on, we will narrow our focus to polynomial rings $R = \mathbb{C}[x_1, \ldots, x_n]$. We can construct a special kind of complex that deconstructs an object piece by piece and analyzes the relationships among those pieces. The following construction is inspired by a clever exposition given by Jason Mccullough and Irena Peeva in [4]. Suppose we have an R-module M that is finitely generated by $m_1, \ldots, m_r$. We call a module a *free module* if it has a basis, that is, if it has a linearly independent generating set. If we're lucky, M will be a free module and $M \cong R^r$. Unfortunately, this is rarely the case but we can always do the following: We can map R^r to M by $e_i \mapsto m_i$ where e_i are basis elements of R^r so we have

$$R^r \xrightarrow{\begin{pmatrix}\text{generators}\\ \text{of M}\end{pmatrix}} M \longrightarrow 0 .$$

As long as M is not free, this first map has a kernel and the elements of that kernel are vectors (n_i) such that $\sum n_i m_i = 0$. It turns out that this kernel will be finitely generated. Now define a new map from R^s to R^r by $f_i \mapsto n_i$ where f_i are basis elements of R^s. We get

$$R^s \xrightarrow{\begin{pmatrix}\text{a generating}\\ \text{system of}\\ \text{relations on}\\ \text{generators of } M\end{pmatrix}} R^r \xrightarrow{\begin{pmatrix}\text{generators}\\ \text{of M}\end{pmatrix}} M \longrightarrow 0 .$$

We can can continue this process indefinitely or until there are no relations among the relations in the previous map at which point will be left with zero map and every module to the left of that will be 0. We see that since each map is defined to be generated the relations on the generators of the next map such that they combine to zero, $\ker(\delta_i) = \operatorname{im}(\delta_{i-1})$ for all i so this complex is exact everywhere and gives a comprehensive image of the structure of M. This special type of complex is called a free resolution and will play a pivotal role in the rest of this paper.

Definition 2.5. A finite *free resolution* of a finitely generated R-module M is a complex of finitely generated free R-modules

$$\mathbf{F} : 0 \longrightarrow F_r \xrightarrow{\delta_r} F_{r-1} \xrightarrow{\delta_{r-1}} \dots \xrightarrow{\delta_1} F_0 \xrightarrow{\delta_0} M \longrightarrow 0.$$

that is exact everywhere. A free module is *minimal* if we choose a minimal set of generators for the kernels that arise. For the remainder of this paper, all resolutions will be assumed to be minimal. We define the *i-th Betti number* of a (minimal) resolution $\mathbf{F}$ to be $\beta_i(\mathbf{F}) = \operatorname{rank}(F_i)$.

Let's break the above construction into more manageable pieces. First, we will take a closer look at the relations that form the maps. We adopt the terminology proposed by David Hilbert and call these relations "syzygies".

Definition 2.6. For an ideal $I = \langle f_1, \dots, f_m \rangle$ in $R = k[x_1, \dots, x_n]$, a first *syzygy* of I is a vector $(c_1, \dots, c_m)^T$ of polynomials such that

$$c_1 f_1 + \cdots + c_m f_m = 0.$$

If $\mathbf{v}_1, \dots, \mathbf{v}_t$ are first syzygies, a second syzygy is a vector $\mathbf{w} = (d_1, \dots, d_t)$ such that

$$\sum_{i=1}^{t} d_i \mathbf{v}_i = 0.$$

A third syzygy is a relation on the second syzygies in the same way and so on.

Example 2.7. Let $R = \mathbb{C}[xy]$. If $I = \langle x^2, xy, y^2 \rangle$, we have

$$y(x^2) - x(xy) + 0(y^2) = x^2y - x^2y = 0$$

$$y^2(x^2) + 0(xy) - x^2(y^2) = x^2y^2 - x^2y^2 = 0$$

$$0(x^2) + y(xy) - x(y^2) = xy^2 - xy^2 = 0.$$

We can express these syzygies as the vectors

$$\mathbf{v}_1 = \begin{pmatrix} y \\ -x \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} y^2 \\ 0 \\ -x^2 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ y \\ -x \end{pmatrix}.$$

We can see that $\mathbf{v}_2 = y\mathbf{v}_1 + x\mathbf{v}_3$ so this syzygy is in the span of $\mathbf{v}_1$ and $\mathbf{v}_3$. In fact, $\mathbf{v}_1$ and $\mathbf{v}_3$ generate all first syzygies of I. There are no second syzygies in this case. To see this, suppose

$$a\mathbf{v}_1 + b\mathbf{v}_3 = a\begin{pmatrix} y \\ -x \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ y \\ -x \end{pmatrix} = 0$$

for a and b but then $ay = -bx = 0$ so $a = b = 0$ so there are no relations between the first syzygies.

The next example shows that we can indeed have nontrivial second syzygies.

Example 2.8. Let $I = \langle wx, wz, xy, yz \rangle \subset \mathbb{C}[w,x,y,z]$ then

$$\begin{aligned} z(wx) - x(wz) &= wxz - wxz = 0 \\ y(wx) - w(xy) &= wxy - wxy = 0 \\ y(wz) - w(yz) &= wyz - wyz = 0 \\ z(xy) - x(yz) &= xyz - xyz = 0. \end{aligned}$$

Our first syzygies are

$$\mathbf{u}_1 = \begin{pmatrix} z \\ -x \\ 0 \\ 0 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} y \\ 0 \\ -w \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} 0 \\ y \\ 0 \\ -w \end{pmatrix}, \mathbf{u}_4 = \begin{pmatrix} 0 \\ 0 \\ z \\ -x \end{pmatrix}.$$

Observe,

$$y\mathbf{u}_1 - z\mathbf{u}_2 + x\mathbf{u}_3 - w\mathbf{u}_4 = 0$$

so we have a second syzygy

$$\mathbf{v} = \begin{pmatrix} y \\ -z \\ x \\ -w \end{pmatrix}.$$

We will see in the following examples that by making syzygies the columns of matrices, we get free modules.

Example 2.9. Let $R = \mathbb{C}[x,y]$ and $I = \langle x^2, xy, y^2 \rangle$ as in Example 2.7 and $M = R/I$ then

$$\mathbf{F}: 0 \longrightarrow R^2 \xrightarrow{\begin{pmatrix} x & 0 \\ -y & x \\ 0 & -y \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} x^2 & xy & y^2 \end{pmatrix}} R \longrightarrow M \longrightarrow 0.$$

By our work in Example 1.7, we see that $H_i(\mathbf{F}) = 0$ so F is a minimal free resolution of M with Betti numbers $\{\beta_i(\mathbf{F})\} = \{1,3,2\}$.

Example 2.10. Now let $R = \mathbb{C}[w,x,y,z]$ and $J = \langle wx, wz, xy, yz \rangle$ as in Example 2.8 and $N = S/J$ then let

$$\mathbf{G}: 0 \longrightarrow R \xrightarrow{\begin{pmatrix} y \\ -z \\ x \\ -w \end{pmatrix}} R^4 \xrightarrow{\begin{pmatrix} z & y & 0 & 0 \\ -x & 0 & y & 0 \\ 0 & -w & 0 & z \\ 0 & 0 & -w & -x \end{pmatrix}} R^4 \xrightarrow{\begin{pmatrix} wx & wz & xy & yz \end{pmatrix}} R \longrightarrow 0\,.$$

This is a free resolution of N with Betti numbers $\{\beta_i(\mathbf{G})\} = \{1,4,4,1\}$.

At this point, the relationship we're establishing between free resolutions, syzygies, and Betti numbers may still seem somewhat nebulous. This relationship was established in a theorem given by David Hilbert in [6] but before we state that, let's begin to develop a rigorous definition of the dimension of an R-module.

Definition 2.11. Let R be a polynomial ring. The *projective dimension* of an R-module M with a minimal free resolution $\mathbf{F}$ is given by

$$\operatorname{pdim}(M) := \max\{i : \beta_i(\mathbf{F}) \neq 0\}.$$

Note that the projective dimension of a module is the length of its minimal free resolution.

Theorem 2.1 (Hilbert's Syzygy Theorem)**.** *Let $R = k[x_1, \ldots, x_N]$, M a finitely generated R-module and $p = \operatorname{pdim}(M)$ then $\operatorname{pdim}(M) \leq N$. This means that every finitely generated module M has a free resolution of length less than or equal to N.*

One can find a proof of this theorem in most algebra books or papers that cover homological algebra cf. Theorem 4.15 in [5]. With this theorem, we can ensure that any module over a polynomial ring will have a finite resolution which will allow us to perform concrete computations and closely inspect the properties of modules and their resolutions without having to worry about the infinite case.

3. GRADING AND BETTI DIAGRAMS

Now we'll introduce the concept of grading which will lead us to a new notion of Betti numbers. Let $R = \mathbb{C}[x_1,\ldots,x_n]$ and R_i be the k-subspace of R spanned by all monomials of degree i. In particular, every $f \in R_i$ is a homogeneous polynomial of degree i. Since any element of R can be written as a finite sum of its homogeneous components, we can express

$$R = \bigoplus_{i\in\mathbb{Z}} R_i$$

where, since if $f \in R_i$ and $g \in R_j$ then $fg \in R_{i+j}$, we have $R_iR_j \subseteq R_{i+j}$. We refer to this decomposition as the *standard grading*.

Similarly, an ideal I is called graded if it has a homogeneous set of generators or, equivalently, if $I = \oplus_{i\in\mathbb{Z}}(S_i \cap I)$ and an R-module M is graded if it can be written as a direct sum of its homogeneous components and if $R_iM_j \subseteq M_{i+j}$ for all $i, j \in \mathbb{Z}$. If R and I are both graded, $M = R/I$ will inherit the grading from R. Finally,

Definition 3.1. Let $d \in \mathbb{Z}$. We call $M(-d)$ the module M *shifted d degrees* so $M(-d)_i = M_{i-d}$.

For instance, $x^2 \in R(-3)$ would have degree 5 since $x^2 \in R_2 = R_{5-3} = R(-3)_5$. In particular, if $R = \mathbb{C}[x,y]$ then $R(-3)_5$ is the $\mathbb{C}$-span of $\{x^2, xy, y^2\}$. We will now give the floor to an example that will elucidate the importance of grading in our context and introduce some more ideas that will be key for the rest of this paper.

Example 3.2. Let $R = \mathbb{C}[x,y]$ and I is the ideal $\langle x^3, x^2y, y^2\rangle$. Note that two of the generators of I have degree 3 and the other has degree 2. Suppose $M = R/I$ then

$$0 \longrightarrow R^2 \xrightarrow{\begin{pmatrix} y & 0 \\ -x & y \\ 0 & -x^2 \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} x^3 & x^2y & y^2 \end{pmatrix}} R \longrightarrow M \longrightarrow 0$$

is a free resolution of M with $\beta(M) = \{1,3,2\}$. However, using the notions introduced above, we can rewrite this resolution as

$$0 \longrightarrow R(-4)^2 \xrightarrow{\begin{pmatrix} y & 0 \\ -x & y \\ 0 & -x^2 \end{pmatrix}} R(-3)^2 \oplus R(-2) \xrightarrow{(x^3\ x^2y\ y^2)} R \longrightarrow M \longrightarrow 0\,.$$

Consider the degree 2 elements of these modules. Since $M = R/\langle x^3, x^2y, y^2\rangle$, M_2 is the $\mathbb{C}$-vector space spanned by $\langle x^2, xy\rangle$. R_2 is simply generated by all degree 2 monomials in x and y and since $R(-2)_2 = R_0$, we get that $[R(-3)^2 \oplus R(-2)]_2 = \langle (0,0,1)^T\rangle$. Finally $R(-4)_2 = R_{-2} = 0$. Let's examine how these elements move through our free resolution, we have

$$0 \longrightarrow 0 \longrightarrow \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle \xrightarrow[(0,0,1)^T \mapsto y^2]{} \langle x^2, xy, y^2\rangle \xrightarrow[\substack{x^2 \mapsto x^2 \\ xy \mapsto xy \\ y^2 \mapsto 0}]{} \langle x^2, xy\rangle \longrightarrow 0\,.$$

For the degree 3 elements we have

$$\cdots 0 \longrightarrow \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle \xrightarrow[\substack{(1,0,0)^T \mapsto x^3 \\ (0,1,0)^T \mapsto x^2y}]{} \langle x^3, x^2y, xy^2, y^3\rangle \xrightarrow[\substack{x^3 \mapsto 0 \\ x^2y \mapsto 0 \\ xy^2 \mapsto xy^2 \\ y^3 \mapsto y^3}]{} \langle xy^2, y^3\rangle \longrightarrow 0$$

and so on for higher degrees. Observe how since, for example $(1,0,0)^T \in [R(-3)^2 \oplus R(-2)]_3 \longmapsto x^3 \in R_3$, the maps in free resolutions like this one preserve degree.

Recall that the Betti numbers β_i of a resolution are the ranks of the modules within it. With this new, graded version of a free resolution, we can define the graded Betti numbers β_{ij}.

Definition 3.3. Let

$$\mathbf{F}: 0 \longrightarrow F_r \xrightarrow{\delta_r} F_{r-1} \xrightarrow{\delta_{r-1}} \cdots \longrightarrow F_0 \xrightarrow{\delta_0} M \longrightarrow 0$$

be a graded free resolution of M. Each F_i is the free resolution of M decomposes as

$$F_i = \bigoplus_i R(-j)^{\beta_{ij}}$$

for some number $\beta_{ij}(M)$. These are called the graded Betti numbers of M.

Note, since each F_i is finitely generated, only finitely many β_{ij} are nonzero and $\sum_j \beta_{ij} = \beta_i$.

Definition 3.4. The *Betti diagram* of a graded free module M is the array given by

$$B(M) = \begin{array}{c|cccc} & F_0 & F_1 & \dots & F_r \\ \hline & \beta_{0,0}(M) & \beta_{1,1}(M) & \dots & \beta_{r,r}(M) \\ & \beta_{0,1}(M) & \beta_{1,2}(M) & \dots & \beta_{r,r+1}(M) \\ & \beta_{0,2}(M) & \beta_{1,3}(M) & \dots & \beta_{r,r+2}(M) \\ & \vdots & \vdots & & \vdots \end{array}$$

where the i,j-th entry is $\beta_{i,i+j}$. Notice how this table is not indexed in the common way that, for example, a matrix would be. Instead, the Betti number that lies in the n-th row and m-th column is $\beta_{m,m+n}$. This way, the degree shift of a module in a free resolution is communicated by the number of steps between $\beta_{0,0}$ and the Betti number corresponding to that module.

If a module has a free resolution of length p then the $p+1$ modules that appear will have nonzero rank. Thus, the first $p+1$ columns of the Betti table will be nonempty with the simplest case being when there is only one Betti number in each column. We give this case a special name.

Definition 3.5. We say that a free resolution **P** is *pure* if its Betti diagram has only one entry per column. In other words, if we can write

$$\mathbf{P} : 0 \longrightarrow R(-d_r)^{\beta_r} \xrightarrow{\delta_r} R(-d_{r-1})^{\beta_{r-1}} \xrightarrow{\delta_{r-1}} \dots \xrightarrow{\delta_1} R(-d_0)^{\beta_0} \xrightarrow{\delta_0} M \longrightarrow 0\ .$$

We call $\mathbf{d} = \{d_0, d_1, \dots, d_r\}$ the *degree sequence* of **P**.

Let's see some examples of Betti tables from resolutions we've all ready explored.

Example 3.6. Let $R = \mathbb{C}[x,y]$, $I = \langle x^2, xy, y^2 \rangle$, and $M = R/I$ as in Example 1.7. The graded version of the free resolution we found before is

$$0 \longrightarrow R(-3)^2 \xrightarrow{\begin{pmatrix} x & 0 \\ -y & x \\ 0 & -y \end{pmatrix}} R(-2)^3 \xrightarrow{(x^2\ xy\ y^2)} R \longrightarrow M \longrightarrow 0.$$

Since there is one generator for $F_0 = R$ of degree 0, we have $\beta_{0,0} = 1$. Likewise, there are 3 generators for F_1 of degree 2 and 2 generators for F_2 of degree 3 so $\beta_{1,2} = 3$ and $\beta_{2,3} = 2$. Which gives the Betti table

$$B(M) = \begin{array}{|ccc} F_0 & F_1 & F_2 \\ \hline 1 & - & - \\ - & 3 & 2 \\ - & - & - \end{array}$$

where the dashes represent zeros. Note that this resolution is pure with degree sequence $\mathbf{d} = \{0,2,3\}$.

Example 3.7. Let $R = k[x,y]$, $I = \langle x^3, x^2y, y^2 \rangle$, and $M = R/I$ as in Example 3.2. We have

$$B(M) = \begin{array}{|ccc} F_0 & F_1 & F_2 \\ \hline 1 & - & - \\ - & 1 & - \\ - & 2 & 2 \\ - & - & - \end{array}.$$

This resolution, on the other hand, is not pure.

4. HILBERT FUNCTION AND SERIES

Let $M = R/\langle x^5, x^2y, y^4 \rangle$. We want to know the number of generators for each graded piece of M. M_2 and M_3Z both have three generators, $\{x^2, xy, y^2\}$ and $\{x^3, xy^2, y^3\}$ respectively. M_4 only has one generator, $\{x^4\}$, and M_i has zero for all $i > 4$. We call the count of these generators the Hilbert function of M.

Definition 4.1. Let M be a finitely generated R-module. The *Hilbert function* of M is given by

$$\varphi_M(d) := \dim(M_d).$$

We call the power series given by

$$Q_M(t) := \sum_{i=0}^{\infty} \varphi_M(i) t^i$$

the *Hilbert series* of M. [1]

Example 4.2. Let $M = \mathbb{C}[x,y]/\langle x^2, xy, y^2\rangle$ as in Examples 1.8 and 2.6. Since $M_0 = \langle 1\rangle$, $M_1 = \langle x, y\rangle$ and $M_k = 0$ for $k > 1$. We have

$$\varphi_M(d) = \{1,2,0,\ldots,0\} \text{ and } Q_M(t) = 1+2t.$$

Example 4.3. Let $N = \mathbb{C}[x,y]/\langle xy\rangle$ then the monomials of degree d are just x^d and y^d so there are two for each degree which gives

$$\varphi_N(d) = \{1,2,2,2,\ldots\} \text{ and } Q_N(t) = 1+\sum_{i=1}^{\infty} 2t^i = \sum_{i=0}^{\infty} 2t^i - 1 = \frac{2}{1-t} - 1.$$

4.1. SOME FACTS ABOUT DIMENSION

Before we go any further, we will establish some useful facts about dimension based on the definition of exactness and on the rank-nullity theorem one encounters in linear algebra.

Lemma 4.1. *Let*

$$0 \longrightarrow U \xrightarrow{f} V \xrightarrow{g} W \longrightarrow 0$$

be a short exact sequence of vector spaces. Then

$$\dim U - \dim V + \dim W = 0.$$

Proof. The rank-nullity theorem gives us that

$$\begin{aligned} \dim U &= \dim(\operatorname{im} f) + \dim(\ker f) \\ \dim V &= \dim(\operatorname{im} g) + \dim(\ker g) \\ \dim W &= \dim W \end{aligned}$$

but since this sequence is exact, we know $\operatorname{im} f = \ker g$. Furthermore, since we know f is injective, $\ker f = 0$ and g is surjective so $\operatorname{im} g = W$. Hence, we have

$$\begin{aligned} \dim U &= \dim(\operatorname{im} f) + 0 \\ \dim V &= \dim W + \dim(\operatorname{im} f) \\ \dim W &= \dim W \end{aligned}$$

so $\dim U - \dim V + \dim W = \dim(\operatorname{im} f) - \dim W - \dim(\operatorname{im} f) + \dim W = 0$. $\square$

Although free resolutions aren't usually short exact sequences, there are short exact sequences hidden within them that will allow us to use this lemma to our advantage.

Lemma 4.2. *Let*

$$\boldsymbol{F} : 0 \longrightarrow F_r \xrightarrow{\delta_r} F_{r-1} \xrightarrow{\delta_{r-1}} \dots \xrightarrow{\delta_1} F_0 \xrightarrow{\delta_0} M \longrightarrow 0$$

be a graded free resolution then $\dim(M_k) = \sum_{i=0}^{r}(-1)^i \dim([F_i]_k)$ *where* M_k *is the k-th graded piece of* M.

Proof. We will take the k-th graded piece of each module F_i. Since our maps have degree zero, the following will be a short exact sequence:

$$0 \longrightarrow [\ker(\delta_0)]_k \hookrightarrow [F_0]_k \xrightarrow{\delta_0} M_k \longrightarrow 0\ .$$

By Lemma 3.1,

$$\dim(M_k) = \dim([F_0]_k) - \dim([\ker \delta_0]_k). \tag{1}$$

Since, $\mathbf{F}_\bullet$ is exact, we know $[\ker \delta_0]_k = [\operatorname{im} \delta_1]_k$ which means that δ_1 is a surjection from $[F_1]_k$ to $[\ker(\delta_0)]_k$ so

$$0 \longrightarrow [\ker(\delta_1)]_k \hookrightarrow [F_1]_k \xrightarrow{\delta_1} [\ker(\delta_0)]_k \longrightarrow 0$$

is a short exact sequence which means $\dim([\ker \delta_0]_k) = \dim([F_1]_k) - \dim([\ker \delta_1]_k)$. Plugging this into (1) gives

$$\dim(M_k) = \dim([F_0]_k) - \dim([F_1]_k) + \dim([\ker(\delta_1)]_k). \tag{2}$$

If we continue in this way across the whole resolution we will get

$$\dim(M_k) = \sum_{i=0}^{r}(-1)^i \dim([F_i]_k).$$

$\square$

Recall, in Example 3.2 that

$$\begin{aligned} \dim([F_0]_2) - \dim([F_1]_2) &= 3 - 1 = 2 = \dim(M_2) \\ \dim([F_0]_3) - \dim([F_1]_3) &= 4 - 2 = 2 = \dim(M_3). \end{aligned}$$

4.2. HILBERT SERIES OF POLYNOMIAL RINGS

Suppose we want to find the Hilbert series of the most basic R-module: R itself. The following Lemma provides an example of this interesting case that will be useful later.

Lemma 4.3. *Let R be a polynomial ring in n variables as usual. Then*

$$Q_R(t) = \frac{1}{(1-t)^n}.$$

Proof. We will proceed by induction. If $n = 1$, we only have one variable so the monomials of R are just powers of that variable so $\varphi_R(i) = 1$ for all i. So

$$Q_R(t) = \sum_{i=0}^{\infty} t^i = \frac{1}{1-t}.$$

Assume this holds through the case where $R = \mathbb{C}[x_1, \ldots, x_n]$. If $n = N+1$, say we adjoin an extra variable t. Note that $\phi_{R[t]}(d) = \sum_{i=0}^{d} \dim_{\mathbb{C}}(R_i)$. To illustrate this fact, let $R = \mathbb{C}[x, y]$. In this case, $\dim_{\mathbb{C}}(R_i)$ is the number of monomials of degree i in 2 variables which will always be $i+1$ (think of Pascal's triangle). Furthermore, $\varphi_{R[t]}(d)$ is the number of degree d monomials in 3 variables which is the $(d+1)$-th triangular number or $\sum_{i=1}^{d+1} i$ so

$$\varphi_{R[t]}(d) = \sum_{i=1}^{d+1} = \sum_{i=0}^{d} i+1 = \sum_{i=0}^{d} \dim_{\mathbb{C}}(R_i).$$

In general, for $R = \mathbb{C}[x_1, \ldots, x_n]$, monomials in $R[t]$ of degree d will be of the form $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} t^b$ where $a_1 + a_2 + \cdots + a_n + b = d$. Observe that if $b = 1$, the number of monomials of degree d is given by $\varphi_R(d-1) = \dim_{\mathbb{C}}(R_{d-1})$ since b will make up for the lost degree. Likewise, if $b = 2$, the number of monomials of degree d is given by $\varphi_R(d-2) = \dim_{\mathbb{C}}(R_{d-2})$ and so on so

$$\varphi_{R[t]}(d) = \sum_{i=0}^{d} \dim_{\mathbb{C}}(R_i).$$

This fact gives

$$
\begin{aligned}
Q_{R[t]}(t) &= \sum_{d=0}^{\infty}\left(\sum_{i=1}^{d}\dim_k(R_i)\right)t^d \\
&= \dim_k(R_0)t^0 \\
&+\dim_k(R_0)t^1+\dim_k(R_1)t^1 \\
&+\dim_k(R_0)t^2+\dim_k(R_1)t^2+\dim_k(R_2)t^2 \\
&+\dim_k(R_0)t^3+\dim_k(R_1)t^3+\dim_k(R_2)t^3+\dim_k(R_3)t^3 \\
&\vdots \\
&= Q_R(t)+Q_R(t)t+Q_R(t)t^2+\dots \\
&= \frac{Q_R(t)}{(1-t)} = \frac{1}{(1-t)^{n+1}}
\end{aligned}
$$

So our claim is true for polynomial rings in any number of variables. □

Example 4.4. Let $R=\mathbb{C}[x,y]$ then $R_0=\langle 1\rangle$, $R_1=\langle x,y\rangle$, $R_2=\langle x^2,xy,y^2\rangle$, $R_3=\langle x^2,x^2y,xy^2,y^3\rangle$, and so on. Observe that $\varphi_R(d)=d+1$ for all dimensions d. Thus,

$$
\begin{aligned}
Q_R(t) &= \textstyle\sum_{n=0}^{\infty}(n+1)t^n \\
&= 1+2t+3t^2+4t^3+\dots \\
&= (1+t+t^2+\dots)+(t+t^2+t^3+\dots)+(t^2+t^3+t^4+\dots)+\dots \\
&= (1+t+t^2+\dots)+t(1+t+t^2+\dots)+t^2(1+t+t^2+\dots)+\dots \\
&= \textstyle\sum_{n=0}^{\infty}t^n+t\sum_{n=0}^{\infty}t^n+t^2\sum_{n=0}^{\infty}t^n+\dots \\
&= \left(\tfrac{1}{1-t}\right)+t\left(\tfrac{1}{1-t}\right)+t^2\left(\tfrac{1}{1-t}\right)+\dots \\
&= \left(\tfrac{1}{1-t}\right)(1+t+t^2+t^3+\dots) \\
&= \left(\tfrac{1}{1-t}\right)\left(\tfrac{1}{1-t}\right) \\
&= \tfrac{1}{(1-t)^2}
\end{aligned}
$$

4.3. A SPECIAL CASE OF THE HILBERT-SERRE THEOREM

We will now show how the Betti numbers of a module relate to its Hilbert series and how they can be used to find the dimension of the module.

Theorem 4.4. *Let $p=\operatorname{pdim}(M)$, then*

$$Q_M(t)=\frac{\sum_{i=0}^{p}(-1)^i\beta_{ij}\,t^j}{(1-t)^n}.$$

Proof. Since $Q_R(t) = \frac{1}{(1-t)^n}$, $Q_{R(-d)}(t) = \frac{t^d}{(1-t)^n}$ since $t^d \in R(-d)$ has degree 0 in R. By Lemmas 4.2 and 4.3, we have

$$Q_M(t) = \sum_i (-1)^i Q_{F_i}(t) = \sum_i (-1)^i Q_{\oplus R(-j)^{\beta_{ij}}}(t) = \sum_i \sum_j (-1)^i \beta_{ij} Q_{R(-j)}(t) = \frac{\sum_{i,j}(-1)^i \beta_{ij} t^j}{(1-t)^n}$$

□

In fact, if we put $Q_M(t)$ is lowest terms by canceling all common factors of $(1-t)$, we get

$$Q_M(t) = \frac{f(t)}{(1-t)^m}$$

for some m. This is a special case of the Hilbert-Serre theorem and the exponent m turns out to be an important invariant of M.

Definition 4.5. If M is a finitely generated module and $Q_M(t) = \frac{f(t)}{(1-t)m}$ as above then the *dimension* of M is given by $\dim(M) = m$.

Example 4.6. Recall the example where $R = \mathbb{C}[x,y]$ and $I = \langle xy, yz, zx \rangle$ from the Introduction. We argued that since the vanishing locus of these polynomials is the x, y, and z axes, that their dimension should be 1. If $M = R/\langle xy, yz, zx \rangle$ then M has the graded free resolution

$$0 \longrightarrow R(-3)^2 \xrightarrow{\begin{pmatrix} z & 0 \\ -x & x \\ 0 & -y \end{pmatrix}} R(-2)^3 \xrightarrow{(xy\ yz\ zx)} R \longrightarrow M \longrightarrow 0\,.$$

Hence, the Betti diagram of M is given by

	F_0	F_1	F_2
	1	–	–
	–	3	2

We have

$$Q_M(t) = \frac{1-3t^2+2t^3}{(1-t)^3} = \frac{1+2t}{(1-t)^1}$$

so by Definition 4.5, $\dim(M) = 1$ as we expected.

Example 4.7. Similarly, we have $I = \langle vw, wx, xy, yz, zv \rangle$ in $R = \mathbb{C}[v,w,x,y,z]$ from Example 1.2. Using the computer program *Macaulay 2*, we find that M has graded free resolution

$$0 \longrightarrow R(-5) \longrightarrow R(-3)^5 \longrightarrow R(-2)^5 \longrightarrow R \longrightarrow M \longrightarrow 0$$

and Betti Table

	F_0	F_1	F_2	F_3
	1	–	–	–
	–	5	5	–
	–	–	–	1

Hence, the Hilbert series of M is given by

$$Q_M(t) = \frac{1-5t^2+5t^3-t^5}{(1-t)^5} = \frac{1+3t+t^2}{(1-t)^2}$$

so $\dim(M) = 2$.

We've established a way to efficiently compute the dimension of afinitely generated R-module while meeting our intuitive expectations. Additionally, we saw in Example 4.7 that we can compute dimension in this way using a computer, which enables us to quickly and efficiently generate data that can strengthen our intuition and lead to new, well-supported conjectures.

5. THE HERZOG-KÜHL EQUATIONS

We will now begin to develop the main theorem of this paper. Not only is it worth studying for its usefulness in the theory of Betti numbers but also for the exciting mechanics of its proof. In order to better understand these mechanics, let's introduce some important concepts from linear algebra.

5.1. SOME KERNELS AND DETERMINANTS

The proof of this theorem is an application of the Vandermonde determinant and some clever tricks involving computing the kernel of a matrix.

Definition 5.1. A *Vandermonde matrix* is a matrix such that the elements in each column are in a geometric progression like

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ a_0 & a_1 & a_2 & \dots & a_n \\ a_0^2 & a_1^2 & a_2^2 & \dots & a_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ a_0^m & a_1^m & a_2^m & \dots & a_n^m \end{pmatrix}.$$

Vandermonde matrices are useful for how often they appear and for the easy computability of their determinants when they're square. For a square Vandermonde matrix

$$A = \begin{pmatrix} 1 & 1 & \dots & 1 \\ a_0 & a_1 & \dots & a_n \\ \vdots & \vdots & & \vdots \\ a_0^n & a_1^n & \dots & a_n^n \end{pmatrix}$$

where $a_i \neq a_j$ when $i \neq j$ we have $\det(A) = \prod_{j>k}(a_j - a_k)$. This fact is proven by induction on n with base case

$$\begin{vmatrix} 1 & 1 \\ a_0 & a_1 \end{vmatrix} = a_1 - a_0$$

an can be found in [7] among many others. As we'll see in the following example, this fact can be used to compute the kernel of a general matrix.

Example 5.2. Let

$$A = \begin{pmatrix} a & -b & c \\ x & -y & z \end{pmatrix}.$$

We want to find v such that $v \in \ker(A)$. Note that

$$\begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix} = a(bz - cy) - b(az - cx) + c(ay - bx) = 0$$

since there is a repeated row. Similarly,

$$\begin{vmatrix} x & y & z \\ a & b & c \\ x & y & z \end{vmatrix} = x(bz - cy) - y(az - cx) + z(ay - bx) = 0$$

so $v = (bz - cy, -(az - cx), ay - bx)^T \in \ker(A)$. This fact extends to a general $n \times (n+1)$ matrix B in which case $(|B_0|, -|B_1|, \dots, \pm|B_n|)^T \in \ker(B)$ where B_i is the $n \times n$ matrix formed by removing the i-th column of B.

We now have all the tools we need to state and prove one of the key equations Jürgen Herzog and Michael Kühl gave in [1] which catalyzed the development of Boij-Söderberg theory.

Definition 5.3. If $\mathbf{d} = \{d_0, \ldots, d_N\}$, then

$$\pi(\mathbf{d}) = \prod_{i \neq j} \frac{1}{|d_i - d_j|}.$$

Theorem 5.1 (Herzog-Kuhl)**.** *If* $\dim(M) = 0$ *and* M *has a pure resolution, then*

$$\beta(M) = \lambda(\pi(\boldsymbol{d}))$$

for some $\lambda \in \mathbb{Q}$.

Proof. Using Theorem 4.4 and Definition 4.5, since $\dim(M) = 0$ and M has a pure resolution with degree sequence $\mathbf{d} = \{d_0, d_1, \ldots, d_n\}$, $(1-t)^n$ is a factor of $\sum(-1)^i \beta_i t^{d_i} = p(t)$ so $p(1) = p'(1) = \cdots = p^{(n-1)}(1) = 0$. Hence, we have the system of equations

$$\begin{array}{rcl}
p(1) & = & \beta_0 - \beta_1 + \cdots \pm \beta_n = 0 \\
p'(1) & = & d_0\beta_0 - d_1\beta_1 + \cdots \pm d_n\beta_n = 0 \\
p''(1) & = & d_0(d_0-1)\beta_0 - d_1(d_1-1)\beta_1 + \cdots \pm d_n(d_n-1)\beta_n = 0 \\
\vdots & \vdots & \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
p^{(n-1)}(1) & = & \left(\prod_{j=0}^{n-1} d_0 - j\right)\beta_0 - \left(\prod_{j=0}^{n-1} d_1 - j\right)\beta_1 + \cdots \pm \left(\prod_{j=0}^{n-1} d_n - j\right)\beta_n = 0
\end{array}$$

which we can write as

$$\begin{pmatrix}
1 & 1 & 1 & \ldots & 1 \\
d_0 & d_1 & d_2 & \ldots & d_n \\
d_0(d_0-1) & d_1(d_1-1) & d_2(d_2-1) & \ldots & d_n(d_n-1) \\
\vdots & \vdots & \vdots & & \vdots \\
\prod_{j=0}^{n-1}(d_0-j) & \prod_{j=0}^{n-1}(d_1-j) & \prod_{j=0}^{n-1}(d_2-j) & \ldots & \prod_{j=0}^{n-1}(d_n-j)
\end{pmatrix}
\begin{pmatrix}
\beta_0 \\ -\beta_1 \\ \beta_2 \\ \vdots \\ \pm\beta_n
\end{pmatrix} = 0.$$

Since the elements of the second row of the matrix on the left are of the form d and the elements of the third row are of the form $d^2 - d$ so performing the elementary row operation of replacing the third row with the sum of the second and third rows gives elements of the third row the form d^2. Likewise, elements of the fourth row have the form $d^3 - 3d^2 + 2d$ so replacing the fourth row with the fourth row plus three times the third row minus two times the second row gives elements of the fourth row the form d^3.

If we continue performing elementary row operations down the left matrix in this way, we can rewrite this system as

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ d_0 & d_1 & d_2 & \dots & d_n \\ d_0^2 & d_1^2 & d_2^2 & \dots & d_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ d_0^{n-1} & d_1^{n-1} & d_2^{n-1} & & d_n^{n-1} \end{pmatrix} \begin{pmatrix} \beta_0 \\ -\beta_1 \\ \beta_2 \\ \vdots \\ \pm\beta_n \end{pmatrix} = 0$$

without changing the solutions since elementary row operations preserve the kernel of a matrix. Note that the matrix on the left is now an $n \times (n+1)$ Vandermonde matrix. We'll call this matrix V. Since V is a Vandermonde matrix where the determinant of any n of its columns is nonzero, $\operatorname{rank}(V) = n$ so by the rank-nullity theorem, $\dim(\ker V) = 1$. This means that for any $v \in \ker(V)$, $\ker(V) = \langle v \rangle$. Note that if we call $\beta = (\beta_0, -\beta_1, \dots, \pm\beta_n)^T$ then $V\beta = 0$ so

$$\beta \in \ker(V) \Rightarrow \ker(V) = \langle \beta \rangle.$$

By Example 4.2, we know that $(|V_0|, |V_1|, \dots, |V_n|)^T \in \ker(V)$ where V_i are the $n \times n$ Vandermonde matrices formed by removing the i-th column from V. Since the kernel of V is generated by β, there exists a constant c such that $c(|V_0|, \dots, |V_n|)^T = \beta$ which mean that $\beta_i = c|V_i|$. Finally, since V_i is an $n \times n$ Vandermonde matrix,

$$\beta_i = |V_i| = c \prod_{j,k \neq i} |d_j - d_k| = c \frac{\prod_{j>k} |d_j - d_k|}{\prod_{i \neq j} |d_j - d_i|} = \frac{\lambda}{\prod_{i \neq j} |d_j - d_i|} = \lambda \pi(\mathbf{d})$$

where $\lambda = c \prod_{j>k} |d_j - d_k|$. □

Let's see this theorem in action with a simple example where we can compute both sides of the equation by hand to show that they will in fact be equal.

Example 5.4. Let M be a module that has a pure resolution with degree sequence $\mathbf{d} = \{0, 2, 3, 5\}$ such that $\dim(M) = 0$ then Lemma 3.2 gives

$$Q_M(t) = \sum_{i=0}^{3} (-1)^i \beta_i t^{d_i} = \beta_0 - \beta_1 t^2 + \beta_2 t^3 - \beta_3 t^5.$$

If we plug in $t = 1$ we get the system of equations

$$\begin{aligned} \beta_0 - \beta_1 + \beta_2 - \beta_3 &= 0 \\ -2\beta_1 + 3\beta_2 - 5\beta_3 &= 0 \\ -2\beta_1 + 6\beta_2 - 20\beta_3 &= 0 \end{aligned}$$

Solving for β_i gives

$$\beta_0 = \beta_3,\ \beta_1 = 5\beta_3,\ \text{and } \beta_2 = 5\beta_3.$$

Since we know Betti numbers have to be integers, we know $\{\beta_i(M)\}$ is at least $\{1,5,5,1\}$ and can be $\{b,5b,5b,b\}$ where $b \in \mathbb{Z}$. Now, Theorem 4.1 tells us that

$$\begin{aligned} \beta_0(M) &= \lambda(\tfrac{1}{2*3*5}) \\ \beta_1(M) &= \lambda(\tfrac{1}{2*1*3}) \\ \beta_2(M) &= \lambda(\tfrac{1}{3*1*2}) \\ \beta_3(M) &= \lambda(\tfrac{1}{5*3*2}) \end{aligned}$$

which is true for $\lambda = 30$ so our theorem works!

It may seem at first like the conditions of purity and dimension zero are fairly strong and that they might limit the usefulness of this theorem. Let's return to our scenario from Example 1.3 to tie together many of the ideas we've introduced, exhibit how the Herzog-Kühl equations are useful even when these conditions aren't satisfied, and give us one final peek into Boij-Söderberg theory.

Example 5.5. Let $R = \mathbb{C}[a,b,c,d,e,f]$ and $I = \langle a^2, b^2, ac+bd, ae+bf \rangle$ then using *Macaulay 2* we see that the graded free resolution of $M = R/I$ is

$$0 \to R(-8) \to R(-7)^6 \to R(-6)^{15} \to R(-5)^{20} \to R(-4)^{13} \to R(-2)^4 \to R \to M \to 0$$

with pure Betti table

F_0	F_1	F_2	F_3	F_4	F_5	F_6
1	–	–	–	–	–	–
–	4	–	–	–	–	–
–	–	13	20	15	6	1

and degree sequence $\mathbf{d} = \{0,2,4,5,6,7,8\}$. The Hilbert series fo M is

$$Q_M(t) = \frac{1 - 4t^2 + 13t^4 - 20t^5 + 15t^6 - 6t^7 + t^8}{(1-t)^6} = \frac{1 + 2t - t^2 - 4t^3 + 6t^4 - 4t^5 + t^6}{(1-t)^4}$$

so $\dim(M) = 4$. This means we can't apply Theorem 5.1 directly. However,

$$
\begin{array}{|ccccccc}
F_0 & F_1 & F_2 & F_3 & F_4 & F_5 & F_6 \\
\hline
1 & - & - & - & - & - & - \\
- & 4 & - & - & - & - & - \\
- & - & 13 & 20 & 15 & 6 & 1
\end{array}
= \frac{1}{35}
\begin{array}{|ccccccc}
 & & & & & & \\
\hline
3 & - & - & - & - & - & - \\
- & 28 & - & - & - & - & - \\
- & - & 210 & 448 & 420 & 192 & 35
\end{array}
$$

$$
+\frac{9}{140}
\begin{array}{|ccccccc}
 & & & & & & \\
\hline
1 & - & - & - & - & - & - \\
- & 7 & - & - & - & - & - \\
- & - & 35 & 56 & 35 & 8 & -
\end{array}
$$

$$
+\frac{3}{20}
\begin{array}{|ccccccc}
 & & & & & & \\
\hline
1 & - & - & - & - & - & - \\
- & 5 & - & - & - & - & - \\
- & - & 15 & 16 & 5 & - & -
\end{array}
$$

$$
+\frac{3}{20}
\begin{array}{|ccccccc}
 & & & & & & \\
\hline
3 & - & - & - & - & - & - \\
- & 10 & - & - & - & - & - \\
- & - & 15 & 8 & - & - & -
\end{array}
$$

$$
+\frac{1}{4}
\begin{array}{|ccccccc}
 & & & & & & \\
\hline
1 & - & - & - & - & - & - \\
- & 2 & - & - & - & - & - \\
- & - & 1 & - & - & - & -
\end{array}
$$

For brevity, let's rewrite this sum as

$$T = \frac{1}{35}T_6 + \frac{9}{140}T_5 + \frac{3}{20}T_4 + \frac{3}{20}T_3 + \frac{1}{4}T_2.$$

Check that T_i is the Betti table given by the Herzog-Kühl equations for the degree sequence $\{d_0, \ldots, d_i\}$. In fact, one of the conjectures proposed by Boij and Söderberg stated that any Betti table can be decomposed into sums of Betti tables given by the Herzog-Kühl equations in this way.

6. CONCLUSION

We've seen that by computing the minimal graded free resolution of a module over a polynomial ring, we can determine the module's Betti numbers. Hilbert and Serre proved that we can use these numbers to uncover more information on the module such as its Hilbert series and dimension. Betti numbers are valuable for this reason but they are also intrinsically interesting to study. As with any newly discovered mathematical object, mathematicians wanted to develop a complete classification of the Betti numbers but this proved to be a daunting task. Instead, we wanted to begin by classifying the Betti numbers up to a scalar multiple. Some useful tools for pursuing this goal are the *Herog-Kühl equations* which were given by Jürgen Herzog and Michael Kühl in Theorem 1 of [1]. These equations allow us to find the Betti table up to scalar multiple of a pure, zero-dimensional graded free module using only its degree sequence.

The Herzog-Kühl equations are a key ingredient in the Boij-Soderberg conjecture that was proven by David Eisenbud and Frank-Olaf Schreyer as Theorems 0.1 and 0.2 of [3] which state that every pure Betti table given by the Herzog-Kühl equations are a multiple of a Betti table for a minimal, zero-dimensional free resolution of a module over a polynomial ring and the Betti table of any finitely generated zero-dimensional graded module over a polynomial ring is a linear combination of pure Betti tables given by the Herzog-Hühl equations.

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