Overview: Throughout the semester there will be several mini-projects that you will complete to get a deeper understanding of the material. In this project, you'll get more practice with quantifiers.

Due Dates: You will be required to complete 4 of these mini projects this semester. I will be flexible with the deadlines, but here the are rules:

- You must submit your first one before spring break (this is designed to get you started on the projects)
- You can only submit one mini-project each week of the semester. (This is designed so that you don't wait until the end of the semester, to do the projects, and also so that I will get a reasonable number of projects each week and so I can grade the projects and give them back to you with feedback.)
- Otherwise I will let you work at your own pace.
- A week starts Sunday at 12:00am and ends Saturday at 11:59pm
- Finals week is included for the mini-projects

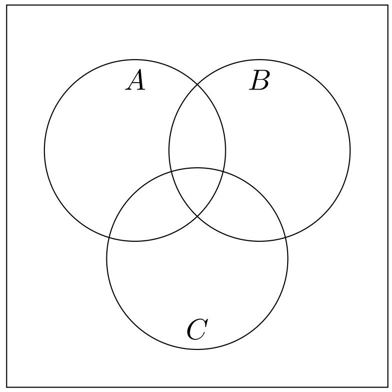
One of my big goals in this class is that we learn to struggle with challenges and push our understanding. I'm happy to help you work on this project in office hours. You are also allowed to work on this project with your peers, but I ask that you write up your solution on your own, in your own words. **You are not allowed to post any part of this project on the internet or seek help from tutors or any websites.** If you are stuck, just let me know.

Contents

1 Understanding Quantifiers

Quantifiers are important because they allow us to make precise mathematical statements. You'll get more and more comfortable with them as the semester progresses. In this mini-project you are going to explore some properties and then determine whether or not certain sets have these properties.

Instructions: Below are 3 properties that a set of real numbers can have, and then 4 sets. Your goal is to **think about which sets have which property**. To help you, you might want to think about the Venn Diagram below. Where would each set go? Does it have property A? property B?



Do I have to do all 4 sets and all 3 properties?

- No - the goal is for you to deeply understand what these properties are saying, and so I am only requiring you to pick two sets and decide whether they satisfy properties A, B, C. (Of course you can do all 4 if you like!)
- So you only have to pick 2 sets, but you need to discuss all three properties for each set.
- In your writeup I want you to explain why the sets you have chosen have or do not have each property.

I will be looking for clear writing that convinces me that you have a solid understanding of the properties and the sets. You do not have to type up your solutions, but you might find that it is easier to type than to write on paper (especially as you begin editing your work). Google drive has an equation editor (just click help and type "equation" and it will activate).

(continued on next page)

The properties:

Note: These properties use the symbol $\in$ which means "in". For instance, the sentence $x \in X$ means "x is in the set X" and $x \in \mathbb{N}$ means "x is in the set of positive integers" i.e. "x is a positive integer."

A) We say that a set X of real numbers has property A if

$$\forall x \in X \ \exists y \in X \ : \ y > x$$

B) We say that a set X of real numbers has property B if

$$\forall x \in \mathbb{N} \ \exists y \in X \ : \ y > x$$

C) We say that a set X of real numbers has property C if

$$\exists y \in \mathbb{N} \ \forall x \in X \ : \ y \leq x$$

The sets:

The open interval $(-\infty, 5)$,

The set of integers $\mathbf{Z}$,

The set of natural numbers $\mathbb{N}$ (i.e. positive integers),

The closed interval $[4, 5]$.

As an example: I will describe how I might write about a hypothetical property D) and some sample sets:

Suppose that a set has property D if:

$$\forall x \in X \ \exists y \in X \ : y > x + 7.$$

1) I will first explain why the open interval $(0, 10)$ does NOT have property D. Property D says that for any x that I choose in X, I should be able to find another y (also in my set X) but with $y > x + 7$. If I choose $x = 9$ then there's no way I can choose a y bigger than $9 + 7$ (which is 15) since my set X only goes up to 10.

A similar argument shows that the intervals $[0, 30]$ and $(-\infty, 100)$ also do NOT have property D. (Since, I could take e.g. $x = 29$ in the first example and $x = 99$ in the second).

For my last two sets, $\mathbb{N}$ and $\mathbf{Z}$ I claim that these DO have property D. Let's see why. No matter what I choose for my x, I will always be able to find a y bigger than $x + 7$ since I have infinitely large numbers to choose from. In case you want me to be more explicit, I could just take $y = x + 100$ which is definitely bigger than $x + 7$ and importantly, if x is in $\mathbb{N}$ then $x + 100$ is still in $\mathbb{N}$ as well. (Ditto for $\mathbf{Z}$).

What do you think - did I convince you?

(end of mini-project)

2 A Proof that $\sqrt{3}$ is irrational.

In this proof you will prove that the $\sqrt{3}$ is irrational. Please read the following carefully and complete the instructions inside of the boxes.

Lemma 1: If n is an integer, then there are only three possibilities for n:

Type 1: $n = 3k$ for some integer k;

Type 2: $n = 3k + 1$ for some integer k;

Type 3: $n = 3k + 2$ for some integer k;

You do NOT need to prove this statement, but it will be useful to prove Lemma 2. Instead, verify it by checking that it is true for the following values of n, and sort them into the three Types:

$$3, 17, -5, -10, 6, 7, 8$$

For each of these numbers, square them and see what Type each number would be in. For instance, 4 is of Type 2 and $4^2 = 16$ is of type 2 as well. Now write a proof of the following statement:

Statement: If n is an integer of Type 3, then n^2 will be of Type 2.

Lemma 2: Let n be an integer. If $3 \mid n^2$ then $3 \mid n$.

Prove Lemma 2 using the cases from Lemma 1.

You will likely be most successful if you use proof by contrapositive.

Think carefully about what the negation of $3|n$ would be in the context of the cases occurring in Lemma 1.

Theorem 3: The square root of 3 is irrational, that is, there is no rational solution to the equation $x^2 = 3$.

Prove Theorem 3 by proof by contradiction.

As always, please write your proofs carefully, neatly and clearly present your work. You are highly encouraged to type your projects, perhaps using Google docs, which has an equation editor.

(end of mini-project)

3 Sometimes / Always / Never

Below are some properties that might be true sometimes, always or never, depending on what the sets A and B are. Your job is to determine which is which.

- In all of these A and B are assumed to be finite nonempty sets.
- Recall that the **powerset** of a set X consists of all subsets of X including the set itself and the empty set.
- For each of the questions below I want you to say whether it is sometimes / always / never true.
- If it is always true, I want you to explain why giving a rough proof. (I will be looking for a solid explanation here).
- If it is sometimes true, then I will be looking for **two** things. First an example when it IS true, and second an example when it is not true. What do I mean by "example" well in this case an example would be a pair of sets A and B. So you might say something like:

  "Example 1: If $A = \{2,3\}$ and $B = \{3,4\}$ then.... and Example 2: If $A = \{...\}$ and $B = \{...\}$ then..."
- If it is never true, then you should give a reason for this. It will not be sufficient to just give one example.

Question 1:

$$\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)?$$

Question 2:

$$\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)?$$

Question 3:

$$\mathcal{P}(A \times B) = \mathcal{P}(A) \times \mathcal{P}(B)?$$

Question 4:

$$\mathcal{P}(A - B) = \mathcal{P}(A) - \mathcal{P}(B)?$$

4 Explaining the Towers of Hanoi

In class you played around with the famous problem: The Towers of Hanoi. In this project, what I would like you to do is explain what you discovered about this problem by preparing a small report about the problem. I will keep things fairly open ended, but here are the main items I will be looking for.

1. I want you to first explain the setup of the problem. What is it we are measuring? In your own words explain what we are looking for. Your goal should be to define the sequence M_n clearly for the reader. You might want to include pictures you can find of this game online.

2. I want you to explain the idea of **recursion** and finding a pattern. Describe what happens in small cases, and include a table of numbers like you did in class.

3. Give your **conjectured recurrence formula**. This is something that is **suggested by the data, but not yet confirmed.**

4. Explain in words **why** this recurrence formula is true in terms of the disks and the towers. (This is the most important - I will want to see that you really understand why this recurrence holds - put this in your own words.)

   Additional Things to Include

   The above are all things I think most groups talked about in their discussions, but here are a few additional problems:

5. Please answer the question given on the Ch. 7 worksheet about "how long until the world ends."

6. In class we will prove that a closed form expression for M_n is $2^n - 1$. Show that this explicit formula satisfies your recursive formula from above.

7. The smallest disk moves a lot more than any of the others. Explore how many steps the smallest disk moves in each case. Let S_n denote how many times the smallest disk must move if there are n disks. Conjecture a recursive formula for S_n and then explain why it is correct. (Again I'm looking for a recursive explanation that relates S_n to S_{n-1} and **why.**

5 Christmas in July

This mini-project expands on the work you did in HW7. Let's start with a true story involving the USD Pool during July 2021. Your professor was swimming laps one day, a very tedious activity. To prevent boredom, he was doing what some people call "pyramids", where he would swim 1 lap, then 2, then 3, then 4, etc. In your homework you learned that the total number of laps swam after such an athletic endeavor is called a Triangular Number, and for instance in the previous sentence he would have swum $T_4 = 10$ laps. You also learned that for $n \geq 1$,

$$T_n = \frac{n(n+1)}{2}.$$

Your professor was pondering these equations while swimming (so much for not being bored!) and noticed the (new-to-him-in-2021) fact that if he added T_3 and T_4 he got a perfect square (in this case 16). And this worked for any two consecutive values of T_n. For instance, $T_2 + T_3 = 9$ and $T_5 + T_4 = 25$.

1. Formulate this statement more generally and write down a statement that captures the general case. Your statement should involve two consecutive terms, and one of those terms should be T_n. The right-hand side of your equation should involve n. State for what values of n your statement is true. Is it true for $n \geq 1$? for $n \geq 0$?
2. Prove this statement in two different ways. First, use the algebraic formula for T_n above to simplify the left hand side and show it is equal to the right hand side.
3. Second prove it by drawing a picture. Your picture should use the "triangular" property of these numbers. I want to see a carefully drawn and labeled picture that shows how "triangles" combine to give a "square". (Imagine your professor's joy at thinking about this picture while swimming laps.)

Now we are going to USE this formula to solve the 12 days of Christmas problem. Remember that on the nth Day of Christmas, T_n gifts are given, and so at the end of the 12 days, there are a total of

$$T_1 + T_2 + \cdots + T_{12}$$

gifts given. Using your results from above, how can you simplify this sum of 12 terms. (Hint: it should simplify down to a sum of 6 terms). Now add up these 6 numbers with a calculator and confirm your result with what you got on HW7.

Finally, I want you to derive a formula for the total number of gifts given on the first $2n$ days of Christmas:

$$C_n = T_1 + T_2 + \cdots + T_{2n-1} + T_{2n}.$$

I want you to use the same method you used above, combining some of the triangular numbers into square numbers.

1) See if you can factor a common factor out of each term.

2) You may use one of the formulas in Project 7 "Some Sums" without proof (you'll be proving that formula there).

You should check that your formula gives the same answer you got in the case of 12 days.

6 Understanding sets and functions:

In this mini-project you will continue to practice your understanding of reading and parsing mathematical statements, but this time it will involve some functions, sets and equivalence relations.

Throughout all of these problems X will be the set consisting of the five sets below:

- The set L consisting of letters in your first name, e.g. my set would be $\{a, d, m\}$.
- The set $M = \{1, 2, 3, 4, 5\}$
- The set $X = \{9\}$
- The set $Y = \mathbb{N}$
- The set $Z = \mathbf{Z}$

You are going to draw 4 **directed graphs.** All graphs will have the same set of **vertices:** L, M, X, Y, Z. I recommend you draw these spaced apart, maybe like a pentagon.

Graph 1: Draw an arrow (directed edge) from vertex A to vertex B if $\exists f : A \to B$, f is injective.

Graph 2: Draw an arrow (directed edge) from vertex A to vertex B if $\forall f : A \to B$, f is injective.

Graph 3: Draw an arrow (directed edge) from vertex A to vertex B if $\exists f : A \to B$, f is surjective.

Graph 4: Draw an arrow (directed edge) from vertex A to vertex B if $\forall f : A \to B$, f is surjective.

For each graph, write a short explanation that captures the salient features of it. For instance, can you explain why you drew in some of the arrows (you don't have to explain everything, but as a rule of thumb, you should explain why you have drawn in some arrows and also why you haven't drawn other arrows. This question is mostly designed to help give you practice thinking about quantifiers, and functions.

7 Some Sums

One of the first proofs that we ever do with induction is the famous sum:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}. \tag{7.1}$$

There are similar formulas for the sum of higher powers as well. Here are the next few:

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}. \tag{7.2}$$

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}. \tag{7.3}$$

Can you guess what comes next? Probably not - although there is a general pattern, it is quite complicated, but the subject of a beautiful story, * In this project I want to show you a way to inductively get the next one of these formulas from the previous one: Recursion!

Part 1: Warmup

As a warmup, prove, using mathematical induction that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

We are going to USE this formula along with equation 7.1 to prove equation 7.3.

Part 2: Gathering Evidence

Notice that if we multiply out the expression $(n+1)^4 - n^4$, we obtain this fairly innocuous looking expression:

$$(n+1)^4 - n^4 = 4n^3 + 6n^2 + 4n + 1.$$

To see what this is saying, I want you to write out the this equation 5 times, starting with $n = 0$ and ending with $n = 4$. Write the equations clearly in a line and line up the corresponding terms. Now, I want you to add up these 5 equations and carefully see what you get on the left hand side. You should have a very simple left hand side; your right hand side should involve sums of powers of n, including some that we have proven, like 7.1 and 7.3, and the sum of cubes that we want to study.

Part 3: Putting it all together

Now, rather than just writing the equation 5 times, what would happen if you wrote it n times. What would you get? Now use this expression and some algebra to derive the formula 7.3.

Part 4: Onwards!

Now, explain how you could extend this to the next case and derive a formula for

$$1^4 + 2^4 + 3^4 + \cdots + n^4.$$

Check your answer with the webpage listed in the footnote, and simplify your answer to match their form.

*see e.g. `https://en.wikipedia.org/wiki/Faulhaber%27s_formula`.

8 So you think you can factor?

As of March 9, 2022 the largest known prime number is:

$$2^{82,589,933} - 1.$$

This is an example of what's called a Mersenne prime, which is a prime of the form $2^n - 1$. In this project you will explore some of the properties of Mersenne Primes.

Part 1: Are they all prime?

Important: Please do this part of the project before reading further. You will be making a conjecture. It's ok if it's not correct, but try your best and write your evidence down clearly.

Do some experiments with numbers of the form $2^n - 1$. Start with $n = 1$ and see if these numbers appear to be sometimes / always / never prime. What values of n (the exponent) seem to give you prime numbers. You can look online for some online prime testing websites. Hint: Make sure you look carefully at $n = 11$.

Write down a conjecture based off of your experiments. What do you think is true? You may want to have multiple parts: If n is prime, do you think $2^n - 1$ is sometimes/always prime? If n is composite?

Part 2: A Warmup with some algebra

Throughout this project you will need to use the following algebra fact:

$$a^n - 1 = (a-1)(a^{n-1} + a^{n-2} + a^{n-3} + \cdots + a^2 + a + 1). \tag{8.1}$$

As a warmup, explain what is going on in the following:

$$x^6 - 1 = ((x^2)^3 - 1) = (x^2 - 1)((x^2)^2 + (x^2) + 1).$$

I want you to explicitly explain what I have done with each equals sign, and if I used the formula (0.1) then explain what my value of n was and what my value of a was. You will need to understand this step deeply to continue so make sure you are ok with this.

Part 3: The Conclusion

Prove the following lemmas. You will need the factorization formula above:

Lemma 1: Let x, n be positive integers. If $x \geq 3$ and $n \geq 2$ then $x^n - 1$ is NEVER prime.

Now using Lemma 1, explain why can conclude Lemma 2. (In other words you should be able to prove Lemma 2 in one or two sentences by using Lemma 1)

Lemma 2: If x, n are positive integers with $n \geq 2$ and $x^n - 1$ is prime, then $x = 2$.

Lemma 3: If n is composite, say $n = \ell k$ then $2^n - 1$ is not prime.

Now explain how your work above allows you to conclude the following:

Theorem If $n \geq 2$ and $x^n - 1$ is prime then actually $x = 2$ and n must be a prime number. So the only primes of the form $x^n - 1$ are actually of the form $2^p - 1$ where p is prime.

9 A Challenging Proof by Induction: Oral Presentation

In this project I will give you a fun and challenging problem to think about. Instead of writing out your solution, for this project you will present your solution to me in office hours. To do so, please send me an email with your possible times (you may also include some times outside of office hours.) Please email me at least 4 days in advance. I will be looking for you to present your **proof by induction** clearly and carefully. If there are any gaps in your proof, you can come back the following week at "resubmit" your presentation. This oral presentation (and any resubmissions) will count as the "one per week" limit for mini-projects. Full disclaimer - this problem is very challenging, but also very rewarding. I think that once you understand how this proof works you will better appreciate mathematical induction.

The Problem:

Let $n \geq 1$ be a natural number. Let $S = \{1, 2, 3, \ldots, 2n\}$ be the set of the first $2n$ integers. Now let A be a subset of S of cardinality $n + 1$. (As it turns out - there are lots of subsets of this size! For instance if $n = 10$ then there are over $160,000$ different possibilities for A) but no matter which of these subsets you choose, you are guaranteed that within the set A, there will be two elements where one divides the other. In other words, A **must** contain an element that is a multiple of another.

Step 1: Play around with this problem. Can you see why it is true for small cases of n? Start with $n = 1$ then $n = 2$. What do you think? Write down some of your early ideas - I'll ask about these during your presentation. Do you have any ideas about how to solve this more generally? Warning - this is a challenging problem!

Step 2: Here are some hints to help coach you through a proof by induction. We will be using strong induction here, including a very clever trick. In Step 1, you will have proven the base case. And so now suppose the induction hypothesis, assuming you have solved the problem for $k = 1, 3, \ldots, n$:

Induction Hypothesis:
P_k: Any subset of size $k + 1$ of $\{1, 2, \ldots, 2k\}$ will have one element that divides another.
Assume $P_1, P_2, \ldots, P_n$ are true.
Goal: Show that P_{n+1} is true.

You will now start to show the inductive step by taking the set $S = \{1, 2, \ldots, 2n, 2n+1, 2n+2\}$ (why?) and a subset A consisting of $n + 2$ elements. (why $n + 2$?). Now you need to complete the proof.

Hints: (more available if you ask)

1. What if all of the elements of A were in $\{1, 2, \ldots, 2n\}$ What would the induction hypothesis say? Explain why you would be done.
2. What if only $n + 1$ of the elements of A were in $\{1, 2, \ldots, 2n\}$? Would you be done then?
3. So if neither of these are true, what elements must be in your set A? (Hint: you should have two ...)

   continuing in this case:

4. So now how many elements do you have in $A \cap \{1, 2, \ldots, 2n\}$. Will the induction hypothesis apply to this set? (no)
5. Explain why if $n + 1$ were in A you would be done.
6. Now if $n + 1$ is not in A, what happens if you consider the set $A \cup \{n + 1\}$? Can you apply the induction hypothesis to this set? What does this mean?
7. How does your proof finish up?

10 Sorting out the Primes

There are infinitely many primes! Let's review the proof that we saw in class (or will see, depending on when you read this!) In this proof we will use the fact that every positive integer greater than 1 has a prime factor.

Theorem: There are infinitely many primes.

Proof. We will argue by contradiction. Suppose that there were only finitely many primes. We will hope for a contradiction. Let's call them $P_1, P_2, P_3, \ldots, P_n$. In other words our list is $2, 3, 5, 7, \ldots, P_n$ and EVERY prime number is somewhere on this list. So now consider the number

$$m = P_1 \cdot P_2 \cdots P_n + 1.$$

Now, if you look at the expression for m, m is NOT divisible by P_i for any i, since if we divide m by any P_i we will always have a remainder of 1. This is a contradiction, since EVERY number greater than 1 must be divisible by some prime. We have found a contradiction, so our proof is complete. □

Ok, so there are infinitely many primes - but what can we say about them? Are they all even? all odd? Oh wait that's an easy one - the only even number that is prime is 2, and so there are infinitely many odd prime numbers.

If you think about it this way, if we made two buckets:

even primes, odd primes

Then one bucket would have 1 element, and the other would have infinitely many.

Let's keep splitting them up. Let's look at the odd primes: $\{3, 5, 7, 11, 13, \ldots\}$. Since these numbers are odd, when we divide them by 4 they will leave a remainder of 1 or 3. (Explain why remainders 0 and 2 are not possible.) In other words, all of these primes must either be of the form $4k + 1$ or $4k + 3$ for some integer k.

Let's make two buckets now for our odd primes:

$$\{\text{primes of the form } 4k + 1\}, \qquad \{\text{primes of the form } 4k + 3\}$$

Step 1: Find a list of the first 30 odd primes and one by one (in order, starting with 3) place them into these two buckets. As you do this, I want you to record how many are in each bucket at each step. For instance, at the first step, I would have 3 in bucket 2, so I would record that as $0, 1$ since nothing is in bucket 1 and there is one number in bucket 2. After putting in the first 5 odd primes, 3,5,7,11,13, I would have $2, 3$ since $5, 13$ would wind up in bucket 1 and $3, 7, 11$ in bucket 2. You may want to write up your results for this in a computer spreadsheet and link to it in your project. This part of the project should have 30 lines.

Questions: What do you notice about your list? What conjectures might you make based on your evidence? Write down a short paragraph based on what you notice and what you wonder.

Step 2: Every odd number is either of the form $4k + 1$ or $4k + 3$ for an integer k. Let's call these buckets: "Type 1" and "Type 3". For example, 15 is Type 3 and 25 is Type 1.

If we multiply odd numbers together we will obtain another odd number. Complete the multiplications below. (You don't need to prove these, as we will do this in class with modular arithmetic)

(Type 1)(Type 1) = [Type 1/Type 3]

(Type 1)(Type 3) = [Type 1/Type 3]

(Type 3)(Type 3) = [Type 1/Type 3]

Question: So if you start with a number of Type 3 (in other words looking at the right hand side), what can you say about its factors? What if you start with a Type 1 number? (These answers should be different). Your answers to this question will be very useful in Step 3. Try to write sentences like: "The only factors possible are blah." or "It is possible that the factors could be blah or blah, but we can't know for certain which one."

Step 3: You should have found a healthy supply of primes in both buckets. It turns out that there are indeed infinitely many primes in each bucket. In this step you will prove that there are infinitely many primes in the set $\{\text{primes of the form } 4k+3\}$, the Type 3 primes. Do this by adapting the proof by contradiction above. Here are some hints:

- As a guide, I want you to consider a number $m = 4P_1 \cdots P_n + 3$. But I want you to decide what the P_i's will be. Do you want to include all the Type 3 primes? Some of them? Do you want to include the Type 1 primes as well?
- Is your number m Type 1 or Type 3?
- Is there any prime in your bucket that can divide into m? Why or why not? (Be careful - is 3 in your bucket? You might want to revise your bucket!)
- You should use your answer to the Question in Step 2 to make a conclusion. Write this step carefully.

Epilogue:

It is actually true that there are infinitely many primes of Type 1 as well. However, if you try to adapt the proof that you wrote in Step 3, you will not succeed, because unfortunately the number

$$4P_1 \cdots P_n + 1$$

will be of Type 1, and as you saw in Step 2, this number might only have factors from the other bucket. And so we won't get a contradiction. The simplest proofs I know that there are infinitely many primes of Type 1 involve some more advanced number theory (a course we are offering next year!) If you are interested, let me know - I could help design a custom mini-project about this if you want. Or we could continue reading about this after the semester ends.

Bonus Epilogue!

You know there are lots of buckets in the world. For instance, we could look at all the odd primes when we divide by 6 and we would see that every prime is gonna be of the form: $6k+1$ or $6k+3$ or $6k+5$. It turns out that all three of these buckets will have infinitely many primes. In fact, there given any "bucket" of the form $Ak+B$ where A and B are relatively prime, there will be infinitely many primes in that bucket. So e.g. there are infinitely many primes of the form: $12n+7$ and here they are:

$$\{7, 19, 31, 43, 67, 79, 103, 127, 139, 151, \ldots\}.$$

You can read more about this here:

`https://en.wikipedia.org/wiki/Dirichlet%27s_theorem_on_arithmetic_progressions`

11 Problems on Chessboards

A chessboard is an 8 by 8 grid of squares, although more generally we might want to play games on a board that is of a different size. In this project we are going to think about what we can do placing pieces on this chess board.

Part 1) The Non-Attacking Queens: In the game of chess, a queen can move in any of the 8 directions - up/down/left/right and in any of the 4 diagonal directions - and they can go any amount of distance. For instance a queen could attack a piece in the same row or same column or same diagonal on the board.

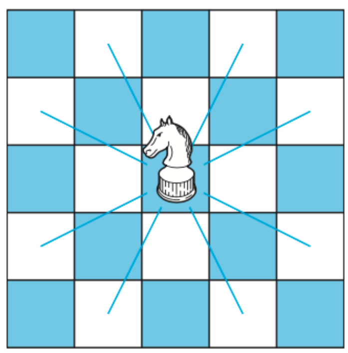
In this project I want you to consider the following question:

"Is it possible to place n queens on an $n \times n$ chessboard in such a way that no queen could attack another? If it is possible, how many different ways are there to arrange the pieces?"

Solve this question for $n = 2, 3, 4, 5$. Look at the number of ways (which is 0 if it's impossible) and now make a guess at to what you think the number will be for $n = 6$. Then work out the case for $n = 6$. Are you surprised by your answer? (Feel free to email me / chat on the discord to check your answers on these.)

For the standard 8 by 8 chessboard, there are 92 ways to put the queens on the board. On a 27 by 27 chessboard, this can be done in 234907967154122528 ways. No one has calculated how many ways there is to do it on a 28 by 28 chessboard. Maybe **you** can!

Part 2) A knight's tour: In the game of chess, a knight can move in the following way: it goes in an L shape that is 1 unit in one direction and then 2 units in a perpendicular direction. For instance the knight in the picture below to move to the indicated squares:



A **knight's tour** is a path that starts somewhere on the chessboard and moves around and visits each square exactly once. We can model this using a graph where each square on the board is a vertex and allowable moves are represented by edges.

1. Draw the graph of allowable moves for a knight on a 3 by 3 chessboard.
2. Draw the graph of allowable moves for a knight on a 3 by 4 chessboard.
3. To find a knight's tour is equivalent to what concept we learned about in class?
4. Show that there IS a knight's tour on a 3×4 chess board. For this, you just need to demonstrate one tour.
5. Show that there IS NOT a knight's tour on a 3×3 chess board. To show this, you'll need to explain why there can't be such a tour.
6. Show that there IS NOT a knight's tour on a 4×4 chess board. To show this, you'll need to explain why there can't be such a tour. (This will feel harder than the previous part, but you can do it. Hint: Think about the corners - and feel free to chat on the discord: `https://tinyurl.com/DiscordMath262Spring2022`
7. Sometimes a knight's tour can be completed in such a way that the knight could move from the end directly back to the start. Explain why such a tour is not possible if the board has an odd number of squares. If you

want to be specific, explain why you can't have such a knight's tour on a 5×11 board. (Hint: think about colors of the squares)