Linear Algebra Materials

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Contents

Part I

Worksheets

1 Welcome to Linear Algebra

You are a young traveler, leaving home for the first time. Your parents want to help you on your journey, so just before your departure, they give you two gifts. Specifically, they give you two forms of transportation: a hover board and a magic carpet. Your parents inform you that both the hover board and the magic carpet have restrictions in how they operate:

We denote the restriction on the hover board's movement by the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.
By this we mean that if the hover board traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 3 miles East and 1 mile North of its starting location. This might be the first time you've seen the word "vector" - that's ok! In this case it just means "direction". Draw a picture of the direction the hover board travels in one hour.

We denote the restriction on the magic carpet's movement by the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.
By this we mean that if the magic carpet traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 1 mile East and 2 miles North of its starting location.

Scenario One: The Maiden Voyage
Your Uncle Cramer suggests that your first adventure should be to go visit the wise man, Old Man Gauss. Uncle Cramer tells you that Old Man Gauss lives in a cabin that is 107 miles East and 64 miles North of your home.

Task:
Investigate whether or not you can use the hover board and the magic carpet to get to Gauss's cabin. If so, how? If it is not possible to get to the cabin with these modes of transportation, why is that the case?

'The Interview' Brings In $15 Million on Web

By MICHAEL CIEPLY DEC. 28, 2014

LOS ANGELES — "The Interview" generated roughly $15 million in online sales and rentals during its first four days of availability, Sony Pictures said on Sunday.

Sony did not say how much of that total represented $6 digital rentals versus $15 sales. The studio said there were about two million transactions over all.

What if anything, would you say to the reporter who reported on this story in 2014?

In the previous tasks you solved a **system of linear equations.** A linear equation is an equation that can be written in the form:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where b and the a_i are real or complex numbers and the x_i are called the **variables.**

A **system of equations** is just a ________________________________

Which of the following is are linear equations? Why?

$y = 3x - \pi$ $\qquad$ $x_1x_2 = x_3$ $\qquad$ $x^2 + y^2 = 4$

Wait, are we really just going to spend a whole semester solving linear equations?

YES - and it's gonna be so much fun!

In high school we often solved systems of 2 equations in 2 unknowns. Something like

$$\begin{cases} -2x + y &= -1 \\ 3x - y &= 3 \end{cases}$$

Geometrically we are ________________________.

What is a **solution** to this **system of equations?**
Do you know different **methods** of solving this system?

Some discussion questions:

A student is solving a system of two linear equations in two unknowns. The student wants to know how many possible **solutions** there are to the system. Can you draw pictures illustrating each of the following possibilities? Hint: Which is the case that we had in the box above?

System has one solution $\qquad$ System has no solutions $\qquad$ System has more than one solution.

Theorem 1: (To be explained next time) A system of linear equations can have:

Before Friday's Class:

- Please read sections 1.1 and 1.2 of the book and complete the MyLab homework assignment.
- In general you will have assignments due each day we have class. I have designed it this way so you will get frequent practice. I will do my best to pick problems that should take somewhere between 30 - 45 minutes.

2 How to Solve a System of Equations

Today we're going to learn how to solve a system of linear equations like:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10 \end{aligned}$$

First things first, we will use a **matrix** to keep track of the **numbers** (this way we don't have to write the variables at each stage of the calculation.)

To the right, let's draw a **coefficient matrix** and **augmented matrix** for this system.

It will be important to remember which is which!

Definition: An $m \times n$ matrix is one with m rows and n columns.

What is the size of the following matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 4 & 6 \\ 1 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 8 & -1 & 4 \end{bmatrix}$$

The matrix C if C is the coefficient matrix for a system of 5 equations in 7 variables.

D is the augmented matrix for a system of 8 equations in 3 variables.

We are going to solve the system on the top of this page together. To do this we are going to use the following three steps

ELEMENTARY ROW OPERATIONS

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.[1]
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a nonzero constant.

- These are things we can do to a matrix.
- With your neighbors, discuss what this would mean for the system of equations.
- Do you think these operations would change anything about the solutions to the system?
- Are these operations "reversible"?

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10 \end{aligned}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$$

There is a video in the "Videos" folder on the google drive in which I work through another example. These calculations aren't the most fun, but it's good to do some to get practice. Once you do a few, you'll get a feel for what's going on. Pretty soon we'll use a computer to do these calculations, but it's important to be able to do this by hand too.

TWO FUNDAMENTAL QUESTIONS ABOUT A LINEAR SYSTEM

1. Is the system consistent; that is, does at least one solution *exist*?
2. If a solution exists, is it the *only* one; that is, is the solution *unique*?

Our general strategy to solve this problem is to:

1) Use ______________________ to simplify the augmented matrix for the system.

2) Use ______________________ to determine if the system has a solution.

We will say a system is ______________________ if there is at least one solution.

If there is no solution we say the system is ______________________.

Some discussion problems:

A student is solving the system:

$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 4x_1 - 8x_2 + 12x_3 &= 1 \end{aligned}$$

and after doing some simplifying of the augmented matrix discovers that this matrix is **row equivalent** to the matrix:

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 15 \end{bmatrix}$$

Is this system consistent? Why or why not?

A student is solving a system of equations ??????? (You don't know the original system) and after doing some simplifying of the augmented matrix discovers that the augmented matrix for their system is **row equivalent** to the matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Is this system consistent? Why or why not?

A student is solving a system of equations ??????? (You don't know the original system) and after doing some simplifying of the augmented matrix discovers that the augmented matrix for their system is **row equivalent** to the matrix:

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Is this system consistent? Why or why not?

- By using **row operations** on some matrix we can modify it to a **row equivalent matrix** that encodes the same system of equations.
- The "simpler" this matrix is, the easier it will be to deduce information about the solutions to the original system of equations.
- Luckily there is a way to do this that is guaranteed to work all the time!

> Theorem (see the textbook for an algorithm) By using row operations, a matrix can always be put into what is called **echelon form.**

A leading entry of a row is the ____________________________.

DEFINITION

A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

EXAMPLE 1 The following matrices are in echelon form. The leading entries ($\blacksquare$) may have any nonzero value; the starred entries ($*$) may have any value (including zero).

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

The following matrices are in reduced echelon form because the leading entries are 1's, and there are 0's below *and above* each leading 1.

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Some (pretty cool facts):

- If a matrix is in REF or RREF, we will often circle the leading entries in each row.

  These are called ____.
- The columns where these pivot positions occur are called _________________________________.
- We will often refer to the pivot columns even of the original matrix. (Don't worry - we'll get used to this)

With your group I want you to talk about which matrices are in **echelon form**. If one is not, talk about why not.

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

What is coming next:

- If you know the echelon form of a matrix, how many solutions does it have?
- If it has a **unique** solution, can we write it down?
- If it has infinitely many solutions, how can we **describe them?**

For instance, what if we are trying to solve a system and we get to this matrix:

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

What to do for Wednesday's class:

- You have an online homework set on MyLab. This is due by 8:00am on Wednesday
- There is a 3×4 matrix that I ask you to put in **reduced** echelon form. I have a video called "HW2 solution" in which I solve this problem.
- There are a few problems in which matrices involve letters like h or f or g. As you are reducing your matrices, you can just think of these as variables that come along for the ride.
- Finally, many of the problems on our assignments will have lots of "multiple choice" problems in which you get multiple attempts at answering the problem. These are designed to help give you practice. If you miss these the first time, you might want to mark them down as problems to study in the future. Note that you may have to click "Try Again" to reset the problem.

3 Reading off Solutions to Equations

Warmup:

This matrix is ALMOST, but not quite in echelon form. Why not? By performing some row operations, put the matrix in echelon form and circle the **pivot positions**.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

This matrix has ____________ pivots

The pivot columns of this matrix are ____________________________________.

With your neighbors discuss the questions below: (T/F means True or False)

1. Every matrix can be put into row echelon form (REF) by using row operations. (T/F)
2. Every matrix can be put into reduced row echelon form (RREF) by using row operations (T/F)
3. The difference between REF and RREF is that in RREF we have to have zeros above the pivots AND

   ____
4. If two students start with a matrix and put it into echelon form, they will always get the same answer no matter what steps they use (T/F)
5. If two students start with a matrix and put it into reduced echelon form, they will always get the same answer no matter what row operations they used to get there (T/F)
6. If two students put a matrix in echelon form the pivot positions can be in different places (T/F)
7. How many solutions does the system below have? What is the echelon form for this system?

$$\begin{cases} x + y &= 8 \\ y &= 7 \end{cases}$$

8. The system of equations below has how many solutions? How would you describe them geometrically? What is the REF for this system?

$$\begin{cases} x + y = 4 \end{cases}$$

9. The system of equations below has how many solutions? How would you describe them geometrically? What is the augmented matrix for this equation and what is its REF?

$$\begin{cases} x + y &= 4 \\ 2x + 2y &= 8 \end{cases}$$

Let's find the general solution to the system of equations given by the augmented matrix in the warmup.

We have seen some examples of systems that have infinitely many solutions. We can **describe** these solutions by identifying what we will call **basic variables** and **free variables.**

A **basic variable** is one that corresponds to a ____

The other variables are called ____.

Let's solve another system and identify the free and basic variables:

$$\begin{cases} x - y + 2z - w &= -1 \\ 2x + y - 2z - 2w &= -2 \\ -x + 2y - 4z + w &= 1 \\ 3x - 3w &= -3 \end{cases}$$

RREF of this system is:

Solutions:

What would the REF have to look like if the system were inconsistent? Draw some pictures

What would it have to look like in order for there to be a **unique** solution?

This is your moment to shine! Your goal is to come up with a way to tell whether a system of equations is consistent. Your answers might involve the words below:

pivot   free variables   basic variables   number of

augmented matrix   echelon form   pivot column   coefficient matrix

First by talking out loud, and then by writing complete sentences

1. When is a linear system consistent/inconsistent?

2. When is a linear system consistent? How can you tell if it has a unique solution? How can you tell if it has infinitely many solutions?

In the next section we will be learning about **vectors**. A vector is a matrix with only one ____________.

For example the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ are both vectors.

The set of all vectors with two rows is denoted $\mathbb{R}^2$.

What do you think $\mathbb{R}^4$ would represent? Can you write down some vectors in $\mathbb{R}^4$?

There are two fundamental things we can do to vectors in $\mathbb{R}^n$:

- We can **add** two vectors to get another vector
- We can **multiply** a vector by a **real number**. This "scales the vector" and we call that real number a "scalar".
- Important point: We never work with vectors of different sizes.
  You CANNOT add a vector from $\mathbb{R}^3$ to a vector in $\mathbb{R}^4$

Examples:

What to do for the next class:

1. As always, there is a short MyLab assignment
2. Read Section 1.3. (As a preview, I will include some short Geometric Problems about vectors on the homework. Examples 1,2,4 in the book will be helpful.)
3. We will have a quiz on Friday. It will be taken from questions from the MyLab Assignments 1,2,3. I might ask you to do some row operations (but nothing too intense). I also might ask questions like "If the augmented matrix for a system of 5 equations in 3 variables has pivots in columns 1 and 3, draw a possible REF matrix for such a system. Does it have a solution, why or why not? If it has a solution, is it unique?

4 What is a Vector?

Vectors are really important for many reasons that will take us all semester to fully explore. For now we're going to get familiar with them in some examples coming from systems of equations.

Example 1:

Write a system of equations that is equivalent to the given vector equation.

$$x_1\begin{bmatrix}5\\-5\\7\end{bmatrix}+x_2\begin{bmatrix}7\\0\\-6\end{bmatrix}=\begin{bmatrix}2\\-3\\6\end{bmatrix}$$

Choose the correct answer below.

A.
$$\begin{aligned}5x_1+7x_2&=2\\-5x_1+x_2&=-3\\7x_1-6x_2&=6\end{aligned}$$

B.
$$\begin{aligned}5x_1+7x_2&=2\\-5x_1&=-3\\7x_1-6x_2&=6\end{aligned}$$

C.
$$\begin{aligned}5x_1+7x_2&=-3\\-5x_1&=2\\7x_1-6x_2&=6\end{aligned}$$

D.
$$\begin{aligned}5x_1+7x_2&=6\\-5x_1&=-3\\7x_1-6x_2&=2\end{aligned}$$

Example 2:

$$x_1\begin{bmatrix}5\\-5\end{bmatrix}+x_2\begin{bmatrix}9\\3\end{bmatrix}+x_3\begin{bmatrix}-2\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

Choose the correct answer below.

A.
$$\begin{aligned}5x_1+9x_2+2x_3&=0\\5x_1+3x_2+x_3&=0\end{aligned}$$

B.
$$\begin{aligned}5x_1+9x_2-2x_3&=0\\-5x_1+3x_2+x_3&=0\end{aligned}$$

C.
$$\begin{aligned}5x_1+9x_2-2x_3&=0\\-5x_1+3x_2&=0\end{aligned}$$

D.
$$\begin{aligned}5x_1+9x_2+2x_3&=0\\5x_1+3x_2&=0\end{aligned}$$

Example 3:

$$\begin{aligned}x_2+4x_3&=0\\5x_1+7x_2-x_3&=0\\-x_1+4x_2-7x_3&=0\end{aligned}$$

Choose the correct answer below.

A.
$$x_1\begin{bmatrix}0\\5\\-1\end{bmatrix}+x_2\begin{bmatrix}1\\7\\4\end{bmatrix}+x_3\begin{bmatrix}0\\0\\0\end{bmatrix}=\begin{bmatrix}4\\-1\\-7\end{bmatrix}$$

B.
$$x_1\begin{bmatrix}0\\5\\-1\end{bmatrix}+x_2\begin{bmatrix}4\\-1\\-7\end{bmatrix}+x_3\begin{bmatrix}1\\7\\4\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

C.
$$x_1\begin{bmatrix}0\\5\\-1\end{bmatrix}+x_2\begin{bmatrix}1\\7\\4\end{bmatrix}+x_3\begin{bmatrix}4\\-1\\-7\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

D.
$$x_1\begin{bmatrix}1\\7\\4\end{bmatrix}+x_2\begin{bmatrix}0\\5\\-1\end{bmatrix}+x_3\begin{bmatrix}4\\-1\\-7\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

A matrix with only one column is called a **vector.** For example, $\begin{bmatrix}2\\5\end{bmatrix}$ is a vector.

- The set of all vectors with two rows is denoted $\mathbb{R}^2$. The $\mathbb{R}$ means that we allow **real number** entries and the 2 means we have two entries.
- In the questions on the left can you find an example of a vector in $\mathbb{R}^3$?

- In the examples on the left you may notice that we are **adding vectors of the same size together** and multiplying vectors by **scalars.**
- For example, calculate

$$2\begin{bmatrix}3\\-1\end{bmatrix}+5\begin{bmatrix}0\\3\end{bmatrix}=\begin{bmatrix}\quad\end{bmatrix}$$

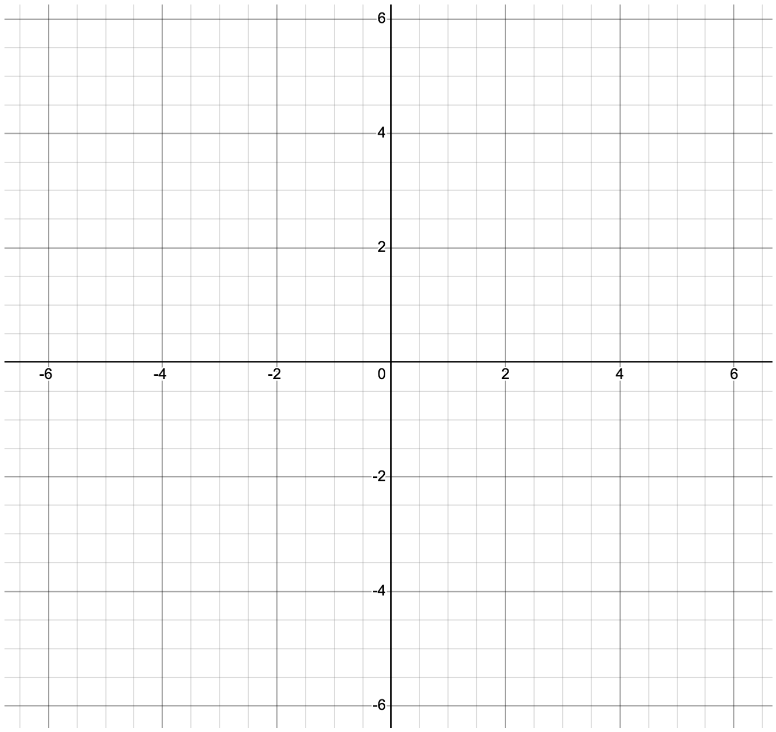
- Vectors can be very useful "shorthand" for writing equations and we will want to develop more of their language together.

- The book will use **bold** letters to denote vectors, like $\mathbf{v}=\begin{bmatrix}1\\4\end{bmatrix}$. When we write by hand we will instead use notation like: ______________________

Remember that to visit Old Man Gauss, you had a magic carpet and hoverboard that could move along vectors

$$\mathbf{m} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \mathbf{h} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Let's draw these in the plane. Where would we draw the vectors $2\mathbf{m}$? What about $\mathbf{m} + \mathbf{h}$? $-3\mathbf{h}$?



Extremely Important Definition:

Let

A **linear combination** of ________________________________ is

Question:

- Which of the vectors in the picture above are linear combinations of $\mathbf{m}$ and $\mathbf{h}$?
- Can you think of a vector that is NOT a linear combination of $\mathbf{m}$ and $\mathbf{h}$?
- Linear combinations of *blah* are the ________________________________

What is wrong with the sentence below:

$$\text{"Decide if } \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \text{ is a linear combination."}$$

Important Point:

We **always** talk about linear combinations ______________________________

What system would we write down to solve this problem from your next homework assignment?

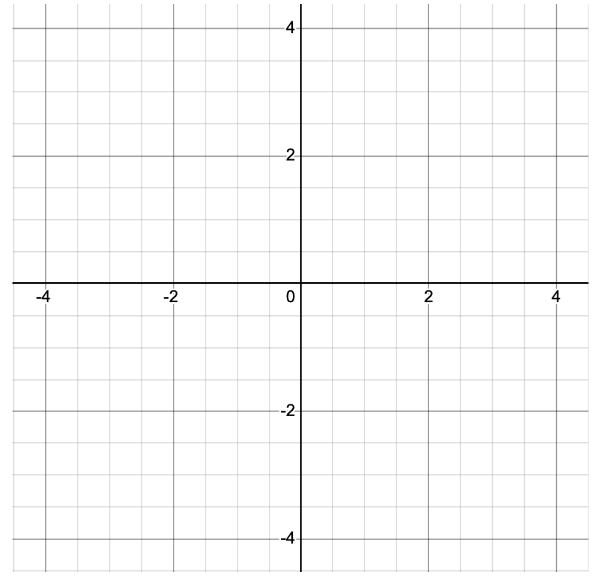
Determine if **b** is a linear combination of the vectors formed from the columns of the matrix A.

$$A = \begin{bmatrix} 1 & -6 & 2 \\ 0 & 3 & 6 \\ -2 & 12 & -4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ -5 \\ -2 \end{bmatrix}$$

Example: Consider the vectors below:

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Sketch them in the graph below:



How would you describe all the vectors that are:

linear combinations of $\mathbf{u}$?

linear combinations of $\mathbf{u}, \mathbf{v}$?

linear combinations of $\mathbf{u}, \mathbf{v}, \mathbf{w}$?

linear combinations of $\mathbf{v}$?

linear combinations of $\mathbf{u}, \mathbf{w}$?

linear combinations of $\mathbf{w}$?

linear combinations of $\mathbf{v}, \mathbf{w}$?

Even More Extremely Important Definition:

Let

The **span** of ______________________________ is

Very often we will ask questions like:

- Is $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ in Span$\{\mathbf{u}, \mathbf{v}\}$?

- Is $\begin{bmatrix} 107 \\ 64 \end{bmatrix}$ in Span$\{\mathbf{m}, \mathbf{h}\}$?

- Is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in the plane spanned by the vectors $\begin{bmatrix} 7 \\ 1 \\ -6 \end{bmatrix}$ and $\begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}$?

5 Old Man Gauss Wants to Hide

Last time we learned about the **span** of a set of vectors. Today we are going to solve some problems about span.

1) Remember that you have a magic carpet and hoverboard that could move along the following vectors.

$$\mathbf{m} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \mathbf{h} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The first day of class you were able to find Old Man Gauss at his current address. However, Old Man Gauss has moved to a new location (somewhere in $\mathbb{R}^2$). Even though you don't know his location, do you think you'll still be able to find him?

- Discuss the question with your group.
- At the whiteboard, can you draw a picture to support your answer
- Also, if Old Man Gauss had moved to location $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, can you set up a system of equations for how to get to his location with **m** and **h**?
- Using what you know about Echelon form and row operations can you help solve for how to get to his new location?

Getting practice reading and understanding definitions:

It's important to be able to read new definitions and understand what they are saying. In the book, the book defines how to multiply a matrix by a vector. Check the definition below and see if you can use it to calculate the problems below it.

DEFINITION

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbb{R}^n$, then the **product of A and x**, denoted by $A\mathbf{x}$, is **the linear combination of the columns of A using the corresponding entries in x as weights**; that is,

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

a. $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$

b. $\begin{bmatrix} 2 & -3 \\ 8 & 0 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix}$

c. If $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are vectors in $\mathbb{R}^m$ then write the linear combination $3\mathbf{v_1} - 5\mathbf{v_2} + 7\mathbf{v_3}$ as a matrix times a vector.

d. On the front page of this worksheet you wrote the vector $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ as a **linear combination** of $\mathbf{m}$ and $\mathbf{h}$. Write this down using a matrix times a vector.

e. Now with your group, read the Theorem below **very slowly** and talk about whether you understand what it is saying. How could you put it into your own words?

THEOREM 3

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$, and if $\mathbf{b}$ is in $\mathbb{R}^m$, the matrix equation

$$A\mathbf{x} = \mathbf{b} \tag{4}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b} \tag{5}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix} \tag{6}$$

Really important punchline:

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if

3D Old Man Gauss

Let $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

Question: Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all choices of b_1, b_2, b_3?

Before we solve this, let's consider what this means geometrically. If the answer is YES, then this means that the

three "vehicles" (the columns of A) would be able to span __________________________.

If the answer is NO, then this would mean there are some places __________________________.

Let's see what the answer is :)

6 Solution Sets of Linear Systems

Question: When does a set of vectors **span** all of $\mathbb{R}^m$?

Last time we saw that the three columns of the matrix A [did / did not] span all of $\mathbb{R}^3$.

$$A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$$

We showed this by showing that it possible for the system

$$A\mathbf{x} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

to be __________________________.
The issue was that the augmented matrix $[\ A \mid \mathbf{b}\ ]$ might have a _______________________________________

Theorem: Let A be an $m \times n$ matrix.

The following are all equivalent:

1) The columns of A span ALL of $\mathbb{R}^m$;

2) Every vector $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A;

3) For any vector $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ __________________________;

4) The matrix A has pivots in _______________________________________.

A system of equations is called **homogeneous** if it can be written in the form $A\mathbf{x} = \mathbf{0}$ for some $m \times n$ matrix A.

Reality Check:

$\mathbf{x}$ is a vector in $[\mathbb{R}^m / \mathbb{R}^n]$ (circle one)

$\mathbf{0}$ is the vector in $[\mathbb{R}^m / \mathbb{R}^n]$ (circle one)

Explain why the equation $A\mathbf{x} = \mathbf{0}$ must ALWAYS be consistent. Explain your answer in at least two different ways.

The solution _____________ is called the **trivial solution.**

It is possible for the equation $A\mathbf{x} = \mathbf{0}$ to have more than one solution.

This will happen if and only if __________________________

If it has more than one solution, it will have __________________________. We call the nonzero solutions **non-trivial solutions.**

EXAMPLE 1 Determine if the following homogeneous system has a nontrivial solution. Then describe the solution set.

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

SOLUTION Let A be the matrix of coefficients of the system and row reduce the augmented matrix $[\,A \quad \mathbf{0}\,]$ to echelon form:

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 2

Describe all solutions to the single equation

$$2x_1 - 5x_2 + 4x_3 = 0.$$

In both of these examples we noticed that the solution set could be described as

Example 3

Let's compare the solutions to the following systems:

System 1:

$2x_1 - 4x_2 = 0$

$-x_1 + 2x_2 = 0$

System 2:

$2x_1 - 4x_2 = 6$

$-x_1 + 2x_2 = -3$

Theorem: Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$. Let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{p} + \mathbf{v}_h$ where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

- Geometrically this is saying that all the solutions to $A\mathbf{x} = \mathbf{b}$ are built from solutions to the homogeneous system $A\mathbf{x} = \mathbf{0}$ by **shifting them** by the vector $\mathbf{p}$.
- Reality check: What would $\mathbf{p}$ be in the example on this page?

Example:

A is ....

	3×3 with 3 pivots	3×4 with 3 pivots	4×3 matrix with 3 pivots
What is the reduced echelon form for A? (Fill in as much as you can)			
Does the system $A\mathbf{x} = \mathbf{0}$ have non-trivial zeros?			
Does the system $A\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ have a solution? How many does it have?			
Explain why we do NOT have enough information to say what those solutions are.			
Do the columns of A span all of $\mathbb{R}^3$?			

Some short answer practice:

1. What is the maximum number of pivots a 7×4 matrix can have?
2. Can a 10×7 matrix have a pivot in every row?
3. Can a 3×2 matrix have a pivot in every column?
4. The columns of a 4×6 matrix are vectors in $[\mathbb{R}^4, \mathbb{R}^6]$ (circle one)
5. In a consistent system of 4 equations in 9 variables, if there are 2 pivot columns in the matrix, then how many free variables are there?

In the next few problems, the system of equations you are considering is the equation

$$A\mathbf{x} = \mathbf{b},$$

so A is the coefficient matrix of a system of equations and B is the augmented matrix for the system.

6. If A has a pivot in the last column then the system is [sometimes/always/never] consistent
7. If B has a pivot in the last column then the system is [sometimes/always/never] consistent
8. If A has a pivot in every row then the system is [sometimes/always/never] consistent
9. If B has a pivot in every row then the system is [sometimes/always/never] consistent
10. If the system is consistent then B [sometimes/always/never] has more pivots than A.
11. If the system is inconsistent then B [sometimes/always/never] has more pivots than A.

7 Getting Back Home

You have just bought three new vehicles at the vector store:

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v_2} = \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}, \quad \mathbf{v_3} = \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}$$

1) You decide to take these vectors out for a spin, and set yourself a challenge!

- You want to ride each vehicle once (either forward or backward), say c_1 hours on $\mathbf{v_1}$, c_2 hours on $\mathbf{v_2}$ etc.
- You want to get back home at the end of your trip.
- You'd ideally like to do this in some **non-trivial** way.

With your group, write down a **vector equation** that sets up the problem you are trying to solve. Then using row operations, determine the answer to the question.

2) (Review from last time) Is there anywhere in this world where Old Man Gauss could hide from you? In other words, is there any location where these vehicles cannot reach? (Hint: you should have some information from your matrix steps above)

In the box below, fill in the left hand side of this equation, using the ingredients you found from above:

$$\mathbf{v_1} + \quad \mathbf{v_2} + \quad \mathbf{v_3} + = \mathbf{0}.$$

From this boxed equation, can you **solve for one of the vectors in terms of the other?** Which one(s) can you solve for?

Important Definition

We say that a set of vectors $\mathbf{v_1}, \ldots, \mathbf{v_p}$ in $\mathbb{R}^n$ is **linearly dependent** if

We say that a set of vectors $\mathbf{v_1}, \ldots, \mathbf{v_p}$ in $\mathbb{R}^n$ is **linearly independent** if

Moral: A set of vectors is **dependent** if one of the vectors can be written as a linear combination of the others. That is, if (at least) one of the vectors is a "redundant vehicle."

Questions: Which of the following sets are linearly independent (LI) or linearly dependent (LD)?

$\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$

$\{\mathbf{v_1}\}$

$\{\mathbf{v_2}\}$

$\{\mathbf{v_3}\}$

$\{\mathbf{v_1}, \mathbf{v_3}\}$

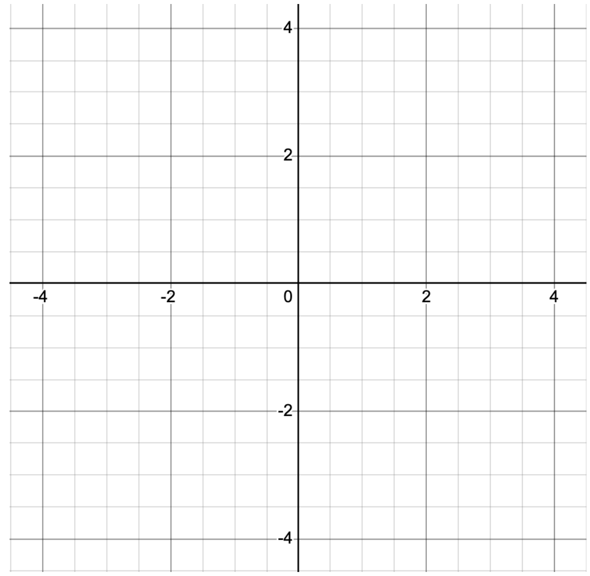
$\{\mathbf{v_1}, \mathbf{v_2}\}$

$\{\mathbf{v_2}, \mathbf{v_3}\}$

Example: Consider the vectors below:

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Sketch them in the graph below:



Is the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ linearly independent?

- Can you solve for one of the vectors in terms of the other? (which one?)

- Can you find a non-trivial combination of these vectors that gives you the zero vector?

Test for Linear Independence

How can we test if a set of vectors is linearly dependent? Let's consider them as the columns of a matrix.

The columns of the matrix are linearly dependent if and only if

Which will happen if and only if

Theorem: The columns of an $n \times m$ matrix A are

- linearly dependent if A has

- linearly independent if A has

8 Getting Practice With Linear Independence

1. When is a set with only one vector $\mathbf{v}$ linearly independent? Remember this would mean that there is a non-trivial solution to

$$c_1\mathbf{v} = \mathbf{0}$$

For instance, which of the sets below would be linearly independent:

$$\left\{\begin{bmatrix}0\\4\\5\end{bmatrix}\right\}, \quad \left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}, \quad \left\{\begin{bmatrix}9\\8\\6\end{bmatrix}\right\}$$

2. The columns of the matrix below are linearly dependent. With your group, without doing any row operations, can you mentally see how to write one vector as a combination of the others? Use this to fill in the blank below.

$$A = \begin{bmatrix}1 & 0 & 1 & -1\\2 & 1 & 3 & 7\\3 & 0 & 3 & 9\\4 & 0 & 4 & 10\\5 & 0 & 5 & 4\end{bmatrix}$$

Mentally I can see that I can write the vector ____________ as a combination of vectors ____________ like this: (fill in the subscripts)

$$\mathbf{v}\ \ =$$

This means that the vectors are linearly [dependent/independent] and my non-trivial solution to $A\mathbf{x} = \mathbf{0}$ will be (fill in the numbers)

$$\begin{bmatrix}1 & 0 & 1 & -1\\2 & 1 & 3 & 7\\3 & 0 & 3 & 9\\4 & 0 & 4 & 10\\5 & 0 & 5 & 4\end{bmatrix}\begin{bmatrix}\ \\ \ \\ \ \\ \ \end{bmatrix} = \mathbf{0}$$

3. To see if the column vectors of a matrix A are linearly independent, you check to see if there is a ____________ in every ____________.

- Use this to quickly write down a matrix whose 4 column vectors form a linearly independent set in $\mathbb{R}^5$.

- Can you write down a matrix whose 5 column vectors form a linearly independent set in $\mathbb{R}^3$. Why or why not?

- What do you think is the largest set of linearly independent vectors you can make in $\mathbb{R}^m$?

4. **Is there a relationship between spanning and linear independence?**

   These are to discuss with your group. I would recommend focusing on listening and talking, rather than writing anything down. Let's imagine you are working in $\mathbb{R}^3$.

   - What is the least number of "vehicles" (vectors) you would need to span all or $\mathbb{R}^3$? Explain your answer.
   - What is the most number of "vehicles" that can form an independent set in $\mathbb{R}^3$?
   - If you know a set of vectors spans all of $\mathbb{R}^3$ does that mean it [sometimes/always/never] is linearly independent?
   - If you know a set of vectors is linearly independent does that mean it [sometimes/always/never] spans all of $\mathbb{R}^3$?

5. You are given a set of 4 vectors in $\mathbb{R}^6$.

   - This set will [sometimes/always/never] span $\mathbb{R}^6$
   - This set will [sometimes/always/never] be linearly independent.

6. You are given a set of 5 vectors in $\mathbb{R}^2$.

   - This set will [sometimes/always/never] span $\mathbb{R}^2$
   - This set will [sometimes/always/never] be linearly independent.

7. You are given a set of 4 vectors in $\mathbb{R}^4$.

   - This set will [sometimes/always/never] span $\mathbb{R}^4$
   - This set will [sometimes/always/never] be linearly independent.

8. Suppose you have 5 vectors in $\mathbb{R}^5$ that span all of $\mathbb{R}^5$. Does it follow that they must be linearly independent? Why or why not?

9. What if you have 5 vectors in $\mathbb{R}^5$ that form a linearly independent set. Does it follow that they must span all of $\mathbb{R}^5$?

10. Fill in the blank: If a $p \times q$ matrix has linearly independent columns then it has ______________ pivots.

11.

   - If B is a 3×2 matrix such that $B\mathbf{x} = \mathbf{0}$ has non-trivial solutions then the columns of B are linearly [dependent/independent/can't tell]
   - If B is a 3×2 matrix such that $B\mathbf{x} = \mathbf{0}$ has a trivial solutions then the columns of B are linearly [dependent/independent/can't tell]
   - If B is a 3×2 matrix such that $B\mathbf{x} = \mathbf{0}$ has only the trivial solution then the columns of B are linearly [dependent/independent/can't tell]
   - If B is a 4×5 matrix such that $B\mathbf{x} = \mathbf{0}$ has a trivial solution then columns of B are linearly [dependent/independent/can't tell]

9 Transformations and Functions

We're going to spend the first 7 minutes of class working together talking about the ideas of linear independence and span.

Recap:

To check if the columns of an $m \times n$ matrix are linearly independent we look for a pivot in every [row/column]

To check if the columns of an $m \times n$ matrix span all of $\mathbb{R}^m$ we look for a pivot in every [row/column]

If the column vectors do NOT span all of $\mathbb{R}^m$ they will still span part of of $\mathbb{R}^m$. The vectors they span can be thought of as the places $\mathbf{b}$ where we can solve the equation $A\mathbf{x} = \mathbf{b}$.

1. How would you best describe the span of a single (non-zero) vector in $\mathbb{R}^3$. Trace out this region with your fingers - showing how you are "spanning."
2. How would you best describe the span of two independent vectors in $\mathbb{R}^3$? Can you use your fingers to point in "two independent" directions. How would your trace out the span of your fingers? Would it be [all/some] of $\mathbb{R}^3$?
3. How would your fingers be if they represented two **dependent** vectors in $\mathbb{R}^3$?
4. Intuitively there are ____________ degrees of freedom in $\mathbb{R}^3$.
5. You are given a set of 4 vectors in $\mathbb{R}^6$.
   - This set will [sometimes/always/never] span $\mathbb{R}^6$
   - This set will [sometimes/always/never] be linearly independent.
6. You are given a set of 5 vectors in $\mathbb{R}^2$.
   - This set will [sometimes/always/never] span $\mathbb{R}^2$
   - This set will [sometimes/always/never] be linearly independent.
7. You are given a set of 4 vectors in $\mathbb{R}^4$.
   - This set will [sometimes/always/never] span $\mathbb{R}^4$
   - This set will [sometimes/always/never] be linearly independent.
8. Suppose you have 5 vectors in $\mathbb{R}^5$ that span all of $\mathbb{R}^5$. Does it follow that they must be linearly independent? Why or why not?
9. What if you have 5 vectors in $\mathbb{R}^5$ that form a linearly independent set. Does it follow that they must span all of $\mathbb{R}^5$?
10. Fill in the blank: If a $p \times q$ matrix has linearly independent columns then it has ____________ pivots.
11.
   - If B is a 3×2 matrix such that $B\mathbf{x} = \mathbf{0}$ has non-trivial solutions then the columns of B are linearly [dependent/independent/can't tell]
   - If B is a 3×2 matrix such that $B\mathbf{x} = \mathbf{0}$ has a trivial solutions then the columns of B are linearly [dependent/independent/can't tell]
   - If B is a 3×2 matrix such that $B\mathbf{x} = \mathbf{0}$ has only the trivial solution then the columns of B are linearly [dependent/independent/can't tell]
   - If B is a 4×5 matrix such that $B\mathbf{x} = \mathbf{0}$ has a trivial solution then columns of B are linearly [dependent/independent/can't tell]

Let's look at the matrix:

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

With your neighbors, calculate the following.

$$\begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \qquad \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \qquad \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} =$$

Sound the fanfare 

Multiplying a vector in __________ by the matrix A will __________ it into a vector in __________.

So far we have thought about matrices as:

- Encoding systems of equations (either augmented or coefficient)
- Representing vectors (as the columns)
- Perhaps as just a collection of numbers (e.g. the pixel color values for an image)

Our new tool is that we can view a matrix as a __________ or sometimes called a __________.

An $m \times n$ matrix will:

"map" or "transform" vectors in __________ and turn them into vectors in __________.

In general when we write a function, we will write it this way:

The set X is called the __________ and represents the allowable inputs.

The set Y is called the __________ and is the set where the outputs must occur.

Example: The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is a function.

- **Every** input has a clear and well-defined output, and the output is in the codomain. For instance, $f(-3) =$ __________
- [Every element / Only some elements] of the codomain actually occur as outputs. For instance the element __________ is in the codomain, but it does not occur as an output.

  This is [ok, and f is still a perfectly fine function. / Not allowed and f isn't actually a function].
- In this case we would say that the __________ of f is equal to the interval __________.
- The __________ doesn't have to be equal to the codomain. (It very often will not be!)

In general, the set of all outputs of a function is called the __________.

If x is an input then we call the output $f(x)$ the __________ of x.

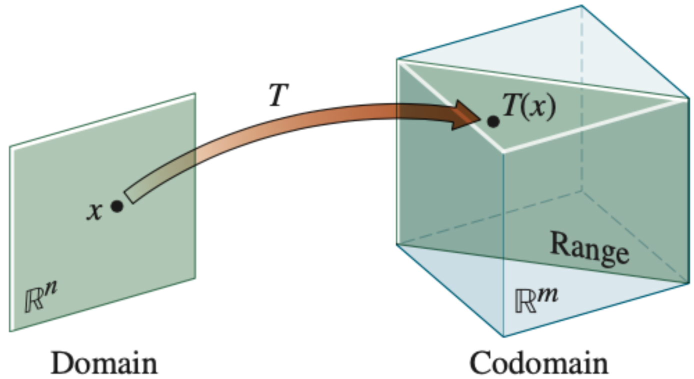


FIGURE 2 Domain, codomain, and range of $T : \mathbb{R}^n \to \mathbb{R}^m$.

Example: The rule $f : \{\text{positive real numbers}\} \to \{\text{positive real numbers}\}$ defined by $f(x) = 2x - 3$ is NOT a function. Why not?

Example: The function $P : \{\text{Math 320 Students}\} \to \{\text{Animals on Earth}\}$ defined by

$$P(x) = x\text{'s pet}$$

is NOT a function. Why not?

I solve this problem below in detail in a video in our videos folder. It is mostly computational. I recommend watching this video as you work through the online HW that is due Friday.

EXAMPLE 1 Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$, and define a transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$, so that

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}$$

a. Find $T(\mathbf{u})$, the image of $\mathbf{u}$ under the transformation T.

b. Find an $\mathbf{x}$ in $\mathbb{R}^2$ whose image under T is $\mathbf{b}$.

c. Is there more than one $\mathbf{x}$ whose image under T is $\mathbf{b}$?

d. Determine if $\mathbf{c}$ is in the range of the transformation T.

Very often we are interested in understanding what a function or transformation does.

Example: Let's see what the transformation defined by

$$T : \mathbb{R}^3 \to \mathbb{R}^3$$

defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

does.

We are trying to figure out what "Multiplying by this matrix" will do to vectors in $\mathbb{R}^3$.

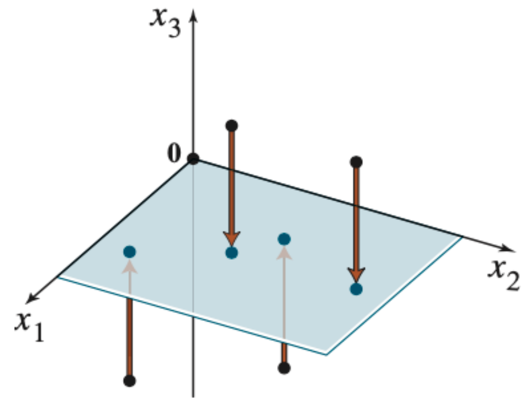


FIGURE 3
A projection transformation.

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

Example

The matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. Defines a transformation

If we applied this transformation to the vectors

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

sheep

sheared sheep

Check out `https://tinyurl.com/LinearTransformationVisualize`

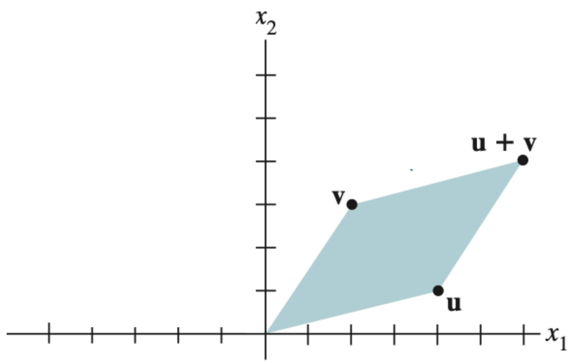
10 Linear Transformations

Warmup before the Quiz:

Define a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T(\mathbf{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$

In the picture below, find the coordinates of the three indicated vectors and **label** them as vectors in the picture. For example $\mathbf{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$. Then plug determine the images of these vectors using the transformation T and draw a picture of the resulting vectors carefully labeling everything. How would you describe what this linear transformation is doing?



On Desmos I have typed in some mystery function $f : \mathbb{R} \to \mathbb{R}$

$$f(3) = 2, \qquad f(5) = -4$$

Do we have enough information to figure out what $f(8)$ is? Why or why not?

Which of the following are true (for all inputs x and y):

- $(x+y)^2 = x^2 + y^2$?
- $e^{x+y} = e^x + e^y$?
- $\ln(x+y) = \ln(x) + \ln(y)$
- $3(x+y) = 3x + 3y$
- $\sin(x+y) = \sin(x) + \sin(y)$
- $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} [x+y] = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} [x] + \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} [y]$.
- $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$.

- $(cx)^2 = c \cdot x^2$
- $e^{cx} = c \cdot e^x$
- $\ln(cx) = c \cdot \ln(x)$
- $3(cx) = c \cdot (3x)$
- $\sin(cx) = c \cdot \sin(x)$
- $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} [cx] = c \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} [x]$.
- $\sqrt{cx} = c \cdot \sqrt{x}$

In general, if $f : \mathbb{R} \to \mathbb{R}$ is a function, it is [______________/always/never] true that

$$f(x+y) = f(x) + f(y)$$

and

$$f(cx) = c \cdot f(x).$$

but it can happen sometimes.

Important Definition

We say that a function $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a ________________________

if the following two properties hold for ____________ ____________ $\mathbf{u}, \mathbf{v}$ in ____________

and all ____________ c:

Example: If we knew that $T : \mathbb{R}^3 \to \mathbb{R}^4$ was a linear transformation and

$$T\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right) = \begin{bmatrix}1\\0\\1\\2\end{bmatrix}, \quad T\left(\begin{bmatrix}1\\0\\4\end{bmatrix}\right) = \begin{bmatrix}1\\2\\0\\1\end{bmatrix}$$

how could we use this information to calculate

$T\left(\begin{bmatrix}2\\2\\7\end{bmatrix}\right)$

$T\left(\begin{bmatrix}3\\0\\12\end{bmatrix}\right)$

After the quiz, please work on the following problems.

You are exploring an unfamiliar territory and you encounter some functions. You are only given partial information about them, but you know that they are **linear transformations**. Luckily you just learned the definition in class and so this will give you a chance to use the definition to see what you figure out.

The first function you find is on the side of the path and it says that

"$f : \mathbb{R}^1 \to \mathbb{R}^1$ is a linear function and $f([2]) = [8]$."

Using property 1 of linearity can you determine what $f([4])$ **must** be? What about $f([-10])$? Are there any values $f([a])$ you would not be able to determine? Be sure to talk with your neighbors when working on this problem and share your ideas with each other.

$f([4]) =$

$f([-10]) =$

Next you encounter a function near the river that says:

"$T : \mathbb{R}^2 \to \mathbb{R}^3$ is a **linear transformation** and " $T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $\quad T(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 8 \\ 0 \\ 5 \end{bmatrix}$"

Can you use the linearity properties to determine the following:

$T(\begin{bmatrix} 6 \\ 0 \end{bmatrix}) =$

$T(\begin{bmatrix} 0 \\ 7 \end{bmatrix}) =$

$T(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) =$

$T(\begin{bmatrix} 0 \\ 0 \end{bmatrix}) =$

Hint: If you are stuck, there is a sign on a stone that says "$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$." Maybe this will help you.

Here are some more

$T(\begin{bmatrix} x_1 \\ 0 \end{bmatrix}) =$

$T(\begin{bmatrix} 0 \\ x_2 \end{bmatrix}) =$

$T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) =$

What do you notice? Can you fill in the blanks with a matrix that gives your answer

$$T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} \quad & \quad \\ \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Let's think about what you just did - you were able to take a very limited information about your transformation. (You only had 2 data points, and the fact that the transformation was linear) and you were able to come up with a formula describing the entire function!

In this example we saw that a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ can be described using a 3×2 matrix.

Some Properties

1. Any linear transformation must satisfy $T(\mathbf{0}) =$ ______________. Why?

2. Any transformation defined by "multiplying by a matrix" is a **linear transformation.**

3. Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ can be described in this way. The matrix that describes this transformation is called the "standard matrix" for T.

For Monday's assignment you will have some problems about "finding the matrix for a linear transformation." The book has several examples of this in Section 1.9. The basic process is like what you did in the previous examples:
For instance if you have a linear transformation: $T : \mathbb{R}^4 \to \mathbb{R}^2$ and you want to find the standard matrix for T then you would be looking for a 2×4 matrix.

The first column of would be $T(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix})$ (which is some vector in $\mathbb{R}^2$)

The second column would be $T(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix})$ (which is some vector in $\mathbb{R}^2$)

the third column would be $T(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix})$ (which is some vector in $\mathbb{R}^2$)

the last column would be $T(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix})$ (which is some vector in $\mathbb{R}^2$)

These very special vectors in $\mathbb{R}^4$ that have only one nonzero entry (and that entry is a 1) are usually denoted by $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, \mathbf{e_4}$

- Based on this what do you think the vector $\mathbf{e}_5$ in $\mathbb{R}^6$ would look like?
- What about the vector $\mathbf{e}_1$ in $\mathbb{R}^2$?

There is a very important (but boring) linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by the rule: $T(\mathbf{v}) = \mathbf{v}$. This is the transformation that doesn't do anything to the vector, it just leaves it alone. We call this vector the **identity transformation.**

Write down the **standard matrix** for the identity transformation $I : \mathbb{R}^3 \to \mathbb{R}^3$.

11 The Geometry of Transformations

Last class we saw that the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ had the effect of "rotating the plane 90 degrees counter-clockwise".

Today we are going to explore some different ways that matrices can **transform** $\mathbb{R}^2$. We also worked out how the **linearity properties** mean that if we understand what $T(\begin{bmatrix} 1 \\ 0 \end{bmatrix})$ and $T(\begin{bmatrix} 0 \\ 1 \end{bmatrix})$ are, then we can use this information to calculate $T(\begin{bmatrix} 6 \\ 0 \end{bmatrix})$ and also $T(\begin{bmatrix} 1 \\ 1 \end{bmatrix})$ and even $T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix})$.

We're going to get practice with this today

Reality Check:

What are the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ in $\mathbb{R}^2$? And how do they look?

A foolproof way to describe the standard matrix for a **linear transformation** $T : \mathbb{R}^2 \to \mathbb{R}^n$:

1. Work out what $T(\mathbf{e}_1)$ is. This will be your first column.
2. Work out what $T(\mathbf{e}_2)$ is. This will be your second column.

This same process would work if we wanted to find the standard matrix for a transformation $T : \mathbb{R}^4 \to \mathbb{R}^2$ all that would be different is that we'd have to work out

$$T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3), T(\mathbf{e}_4)$$

and put those columns into the matrix. (This is what you did in the online HW for today)

Warning: This process only works if you know your function is **linear**.

For instance, let's try a few of these matrices below and see what they do:

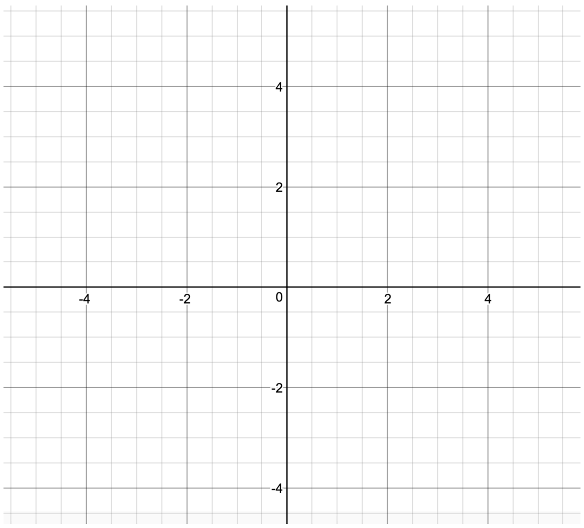
$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

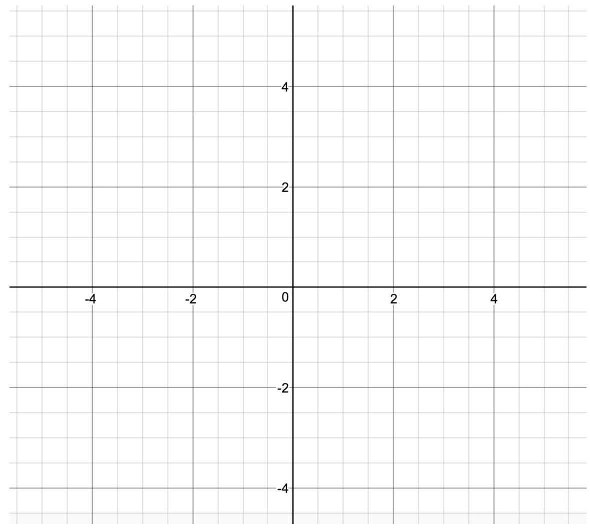
For help visualizing, we will use:

`https://web.ma.utexas.edu/users/ysulyma/matrix/`

Match the transformations below with the matrices that determine them via

$$T : \mathbb{R}^2 \to \mathbb{R}^2, \qquad T(\mathbf{x}) = A\mathbf{x}$$





Rotation by 90 degree counter clockwise

Rotation by 180 degrees

Reflection across the x-axis

Reflection across the line $y = x$

The **identity** transformation that does nothing

Reflection across the y-axis

Stretch in the x direction by a factor of 2

Project onto the y-axis

Project onto the x-axis

Question: What is the standard matrix for the identity transformation on $\mathbb{R}^4$?

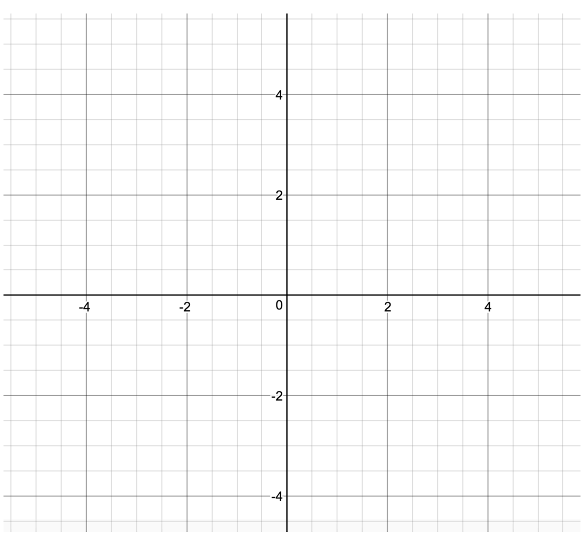
It turns out that all of the following are examples of linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$:

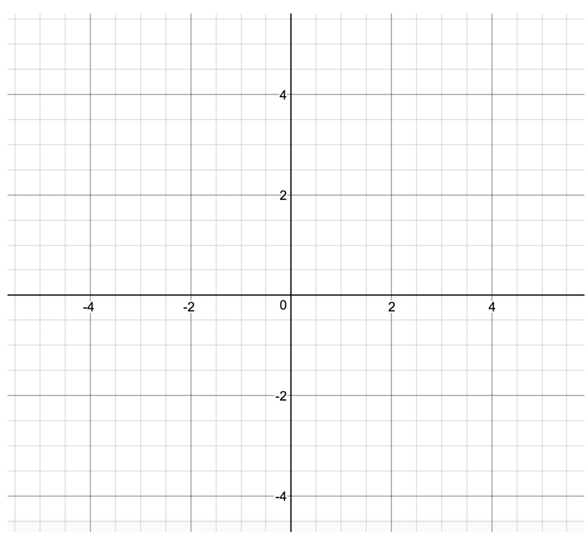
- rotation about the origin counterclockwise by θ degrees:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

- reflection about a line through the origin
- scaling in the x and y direction
- projection onto a line through the origin
- combining or composing any of these operations. (Stay tuned for lots more about composition!)

Example: Write down the matrix for the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that does the following: "Rotate 90 degrees counter-clockwise, then reflect across the x-axis. Then scale by 2 in the x direction."





What about....

In all the previous examples, our process for checking that something was linear was basically:

"Try to write down a matrix that describes the transformation.

If we can do it and that matrix describes our transformation,

then our transformation must be linear, because multiplying by a matrix

definitely has Properties 1 and 2 in the definition of Linear Transformation."

Question: How would you show that something is NOT a linear transformation?

Example: Show that the function $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by $T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} x_1 + 3 \\ x_2 \\ 0 \end{bmatrix}$ is NOT a linear transformation.

To show that something is NOT a linear transformation, we will have to show

-

There is a quicker way that **sometimes** works to show that a function is NOT a linear transformation.

Since any linear transformation must satisfy that ____________ = ____________ if we test and notice that

then that means T is ________________________.

Example: Decide if the functions below are linear or not. If they are linear then determine the standard matrix. If they are not linear, then explain why by giving an explicit violation

$T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} \min(x_1, x_2) \\ x_1 \end{bmatrix}$

$S : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}) = \begin{bmatrix} 3x_1 + x_3 \\ -x_2 - x_3 \\ -4x_1 \end{bmatrix}$

$U : \mathbb{R}^2 \to \mathbb{R}^3$ given by $T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} 6 + x_1 \\ x_2 + x_1 \\ 0 \end{bmatrix}$

Let's think about the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

This is [linear/not linear] because it is defined by ____.

Think about where it will send the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$.

- Geometrically this transformation is:
- Do you think we could transform an input and have it go to the vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$?
- What vectors will be in the **range** of this transformation?
- Does the range of this transformation equal the codomain?
- True or False: There are some vectors $\mathbf{b}$ we cannot get to using

$$A\mathbf{x} = \mathbf{b}.$$

Important Definition

We say that a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called ______________________ if

equivalently, this is the same as saying that the equation

has

Example: How would we figure out if the transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ defined by the matrix

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

is **onto**?

To check if a linear transformation is onto - we can check:

Important Definition

We say that a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called __________________________ if

equivalently, this is the same as saying that the equation

has

Example: How would we figure out if the transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ defined by the matrix

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

is **1-1**?

To check if a linear transformation is 1-1 - we can check:

12 1-1 and Onto Properties

Special Offer!

If you thought magic carpets and hoverboards were cool - just wait till you hear about the new transformations we have for sale! Today for sale I have a beautiful brand new **linear transformation**. $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by multiplication by the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

If you are located at the vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ then this transformation will immediately whisk you away to the new location... wait what was that new location? (Hmm, can you help me out here... multiply by the matrix eh? ...)

$$T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} \quad \end{bmatrix}.$$

Oh thank you! Yes this linear transformation can be yours for just a small price of ....

- Geometrically, what is going on with this transformation? How is it "transforming" the x and y coordinates.
- Do you think we could transform an input and have it go to the vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$? What about $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$?
- What vectors will be in the **range** of this transformation? Remember the range consists of the outputs that "actually occur"
- Does the range of this transformation equal the codomain?
- To see if the vector $\mathbf{b} \in \mathbb{R}^2$ is in the range of this transformation we check whether the system $A\mathbf{x} = \mathbf{b}$ is ________________________.

Reality Check:

- In this example we saw that it is possible that ________________________ vector in the codomain will be realized as an output of the function.
- This means that the ________________________ and the ________________________ are different.
- There are some vectors $\mathbf{b}$ in the codomain for which $T(\mathbf{x}) = \mathbf{b}$ has ____________ solutions.

Important Definition

We say that a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called ________________________ if

equivalently, this is the same as saying that the equation

has

Discuss with your neighbor: Suppose that T is given by multiplication by a matrix A.

Question: Which of the following are **equivalent** to the onto property. This means that it captures exactly what the definition is saying.

1. For every $\mathbf{b}$ in $\mathbb{R}^m$ there is a solution to $A\mathbf{x} = \mathbf{b}$.
2. For some $\mathbf{b}$ in $\mathbb{R}^m$ the system $A\mathbf{x} = \mathbf{b}$ is consistent.
3. Every $\mathbf{b} \in \mathbb{R}^m$ is a linear combination of the columns of A.
4. Every input $\mathbf{x}$ goes to an output vector $\mathbf{b}$.
5. No matter what vector $\mathbf{b} \in \mathbb{R}^m$ Old Man Gauss moves to, we can solve for $\mathbf{x}$ in $A\mathbf{x} = \mathbf{b}$.
6. The columns of A are linearly independent.
7. The columns of A span all of $\mathbb{R}^m$.

Example: How would we figure out if the transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ defined by the matrix

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

is **onto**?

To check if a linear transformation is onto - we can check:

Mild Warning:

- Although essentially **EVERY** transformation we will study in this class will be linear, remember that this property is generally very rare.
- For instance, if we were studying the function

$$f : \{\text{USD students in Math 320}\} \to \{\text{366 days on the calendar}\}, \quad f(x) = x\text{'s birthday}.$$

  what would we need to do to check if F were onto, i.e. that the range of F were equal to the codomain?
- What about

$$g : \{\text{Students worldwide in Linear Algebra}\} \to \{\text{366 days on the calendar}, \quad g(x) = x\text{'s birthday}\}$$

- What about

$$h : \mathbb{R} \to \mathbb{R}, \quad h(x) = \sin x.$$

Hey folks, it's just me - your friendly transformation salesperson!

I hear you sent in a complaint about the transformation, saying that it was not... what was the word? You learned that the transformation was not __________________________. Well you know we have no refunds, I'm afraid to say. But let me remind you again about all the places you CAN go. The x-axis is a wonderful place, and hey don't you like ....

This salesperson sounds kind of verbose, so let's move on to something different. Let's use the same example from before: $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by multiplication by the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, that is:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}.$$

Remember this this transformation, geometrically is **projecting** onto the x-axis.

There are some outputs we cannot reach. (E.g. $T(\mathbf{x}) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ has no solution).

But sometimes we DO have solutions. (E.g. $T(\mathbf{x}) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ does have a solution.)

How many solutions does it have?

Reality Check:

- It can happen that __________________________ inputs will go to the _____________ output.
- This means that there are vectors $\mathbf{b} \in \mathbb{R}^2$ so that $A\mathbf{x} = \mathbf{b}$ has __________________________ solutions.

Important Definition

We say that a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called __________________________ if

equivalently, this is the same as saying that the equation

has

1. For every $\mathbf{b}$ in $\mathbb{R}^m$ there is a solution to $A\mathbf{x} = \mathbf{b}$ and that solution is unique.
2. If there IS a solution to $A\mathbf{x} = \mathbf{b}$ then that solution will be unique.
3. Every input $\mathbf{x}$ goes to an output vector $\mathbf{b}$.
4. There might not always be a solution to $A\mathbf{x} = \mathbf{b}$, but when there is, there are no free variables.
5. If Old Man Gauss lives in a location $\mathbf{b}$ then, by using the "vehicles" that are the columns of A we can always get there in a unique way.
6. If Old Man Gauss lives in a location $\mathbf{b}$ then, by using the "vehicles" that are the columns of A we can get there in either 0 ways or 1 way, but not infinitely many ways.
7. The columns of A are linearly independent.
8. The columns of A span all of $\mathbb{R}^m$.

Example: How would we figure out if the transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ defined by the matrix

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

is **1-1**?

To check if a linear transformation is 1-1 - we can check:

Mild Warning:

- Although essentially **EVERY** transformation we will study in this class will be linear, remember that this property is generally very rare.
- For instance, if we were studying the function

$$f : \{\text{USD students in Math 320}\} \to \{\text{366 days on the calendar}\}, \quad f(x) = x\text{'s birthday}.$$

  what would we need to do to check if F were 1-1, i.e. that the range of F were equal to the codomain?
- What about

$$g : \{\text{Students worldwide in Linear Algebra}\} \to \{\text{366 days on the calendar}, \quad g(x) = x\text{'s birthday}\}$$

- What about

$$h : \mathbb{R} \to \mathbb{R}, \quad h(x) = x^2$$

Final Thoughts:

The onto property and the 1-1 property are best thought to be completely unrelated.

In general it is possible for a function to have one property and not the other, or both or none.

For linear transformations, since we can measure these properties with **pivots** this means that solving questions like:

"The linear transformation defined by a 4×7 matrix [sometimes/always/never] defines a 1-1 transformation."

Is going to be the same as asking:

"The columns of a 4×7 matrix are [sometimes/always/never] ____."

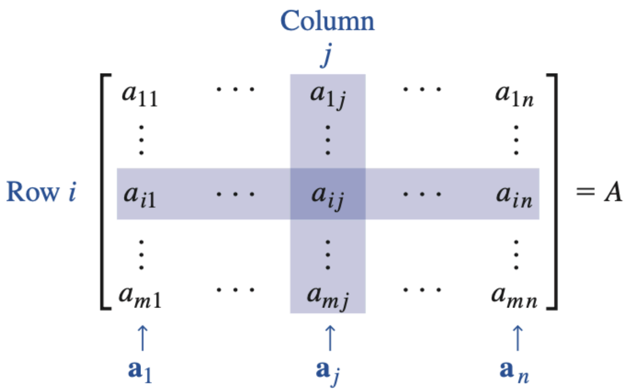
As always, I highly recommended getting some flashcards to keep track of all the different properties, we've been learning, their definitions, tests, etc.

Friday's Quiz will be all about Linear Transformations!

13 Matrix Operations

We are now going to think about different ways to combine and modify matrices.

Warmup (3 minutes + 2 recap)



Sometimes we will want to talk about say the "Entry of the matrix A in the 4th row and 3rd column. We will write that as a_{43}.

As a warmup (and practice for the quiz), write down the standard matrix for the linear transformation that rotates $\mathbb{R}^2$ by 90 degrees CLOCKWISE.

1. In that matrix, what is a_{21}?
2. Is the transformation $1-1$? onto? Why?

Definition of Matrix Addition and Scalar Multiplication

Let A and B be matrices and let r be a real number.

(Scalar Multiplication) Then rA is **always** defined. To calculate rA we just multiply every entry of A by r.

(Addition of Matrices) The sum $A+B$ is **only sometimes defined**. It is defined when A and B are the same size. When A and B are the same size, we will define $A+B$ by adding the corresponding entries in the matrix. (The difference is similarly defined)

Using the matrices below, calculate the expressions, or say whether they are undefined. I_n will denote the $n \times n$ **identity matrix**.

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

$A+B$ $\qquad$ $A+D$ $\qquad$ $3A-B$ $\qquad$ $D-I_2$

The theorem below shows some of the properties that **addition and scalar multiplication** have. (How excited are you by these properties?)

THEOREM 1 Let A, B, and C be matrices of the same size, and let r and s be scalars.

a. $A+B=B+A$
b. $(A+B)+C=A+(B+C)$
c. $A+0=A$
d. $r(A+B)=rA+rB$
e. $(r+s)A=rA+sA$
f. $r(sA)=(rs)A$

Matrix Multiplication

Let $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$. We have learned how to multiply A by a vector. Let's do it 3 times in a row.

$$\begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} \qquad \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

We have just performed the calculations that are necessary to calculate the product:

$$\begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} =$$

This is the "nitty gritty how we calculate stuff" way of multiplying matrices together."

- It is important to be able to do this confidently and quickly.
- It is important to know when matrix multiplication is defined.
- It is important to know the relationship between matrix multiplication and **composition of functions.**

When is matrix multiplication defined

Let's think about it:

Which of these are defined:

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 3 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 3 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 3 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

So if we want to multiply:

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 3 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

What about if we wanted to multiply on the other side?

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 3 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Let A and B be matrices. The product AB is defined only if the number of _____________ in A is equal to the number of _____________ in B.

$$(m \times n)(n \times p) = m \times p$$

Reality Check:
It can happen that AB is defined but BA is not defined (True / False)

Which products are defined?

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

AC $\quad$ CD $\quad$ CB $\quad$ EB $\quad$ BC

What is really going on?

Two computer science majors each write a computer function.

Alice's function

$$A : \{\text{integers}\} \to \{\text{text strings}\}$$

takes in a number and turns it into a string of letters. We're not sure exactly how it works, but for example, if you calculate $A(3)$ it says "dog" and if you said $A(13)$ is says "spooky szn".

Bob's function

$$B : \{\text{text strings}\} \to \{\text{text strings}\}$$

takes as input a string and outputs another text string. Bob says that his code just adds "are cool!" to the string. For instance, if we calculate $B(''dogs'')$ the output is "dogsare cool!".

Question: Which of the following make sense:

$$A(B(''dogs'')), \qquad B(A(3)), \qquad B(B(''dogs'')), \qquad A(A(3))$$

For the ones that make sense, which function is happening "first" the inside function or the outside function?

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 3 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & 2 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 1 & 7 & 0 \\ 0 & 2 & -1 & -1 \end{bmatrix}$$

Matrix Multiplication Represents The Linear Transformation When we Compose Functions

Biggest Warning of the Class (so far):

On the front page you calculated that the matrix for "Rotate 90 degrees clockwise" was

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad\qquad B =$$

Let B be the matrix for "reflect across the y axis.

(Remember, to do this, you just need to see where $\mathbf{e}_1$ and $\mathbf{e}_2$ go.)

Now let's do some calculations:

"First rotate 90 degrees clockwise then reflect across the y-axis"

Doing it geometrically by thinking about where $\mathbf{e}_1$ and $\mathbf{e}_2$ go:

By multiplying matrices:

Did it work? Did the composition agree with matrix multiplication? [Yep / Nah]

"First reflect across the y-axis and then rotate 90 degrees clockwise then"

Doing it geometrically by thinking about where $\mathbf{e}_1$ and $\mathbf{e}_2$ go:

By multiplying matrices:

Did it work? Did the composition agree with matrix multiplication? [Yep / Nah]

14 The Inverse of a Matrix

Warmup:

Take your quiz (or your phone) and hold it facing you. We are going to do the following operations and see if they "commute"

Flip across the horizontal axis THEN rotate 90 degrees counterclockwise.

Flip rotate 90 degrees counterclockwise THEN Flip across the horizontal axis

Did you get the same result?

Biggest Warning of the Class (so far):

On the last day you calculated that the matrix for "Rotate 90 degrees clockwise" was

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad B =$$

Let B be the matrix for "reflect across the y axis.

(Remember, to do this, you just need to see where $\mathbf{e}_1$ and $\mathbf{e}_2$ go.)

Now let's do some calculations:

"First rotate 90 degrees clockwise then reflect across the y-axis"

Doing it geometrically by thinking about where $\mathbf{e}_1$ and $\mathbf{e}_2$ go:

By multiplying matrices:

Did it work? Did the composition agree with matrix multiplication? [Yep / Nah]

"First reflect across the y-axis and then rotate 90 degrees clockwise then"

Doing it geometrically by thinking about where $\mathbf{e}_1$ and $\mathbf{e}_2$ go:

By multiplying matrices:

Did it work? Did the composition agree with matrix multiplication? [Yep / Nah]

Discussion Questions: For all of these questions, your answer should be a "description" of a transformation.

Don't worry about matrices for the moment.

1. If F is a linear transformation that rotates the plane 20 degrees counterclockwise. What would the transformation $T \cdot T$ do? What about T^{10} (do it ten times)?
2. Remember that the identity matrix, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ represents the "do nothing transformation. It sends every vector to itself. What would I^2 be?
3. Chose your favorite linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$. What would happen if you "first did T and then did nothing."
4. What if you "first did nothing and then did T."
5. Relate your two answers above to the equations below

$$TI = T, \quad IT = T.$$

6. Suppose that R is a rotation by 90 degrees counterclockwise. How could you "undo" the transformation R?
7. How would you undo the transformation Y that reflects across the x axis?
8. What about the transformation S that scales the x axis by 3 and the y axis by 4. How would you undo this transformation?
9. What about the transformation P that projects onto the x axis? Could you undo this one? If not, why do you say that?

$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (rotate 90 degrees counterclockwise)

$B = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$ (rotate 90 degrees clockwise)

$C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (reflect across x axis)

$D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ (reflect across y axis)

$E = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ (scale x by 3 and y by 4)

$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ (project onto the x axis)

Let's write down some equations based on what we discovered above.

Very Important Definition

An ____________ matrix A is called __________________ if

Facts:

- It is possible that A does not have an inverse;
- If A **does** have an inverse, then the inverse is unique;
  In this case, we will denote the inverse by A^{-1}.

Continuing with the previous problem, let's find the inverses of the matrices we saw.

Example:

$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (rotate 90 degrees counterclockwise)

We saw that the matrix for "rotate 90 degrees clockwise" was the inverse of A.

$A^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

EXAMPLE 1 If $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ and $C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$, then

$$AC = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}\begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \qquad\qquad \text{and}$$

$$CA = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} =$$

Questions:

1. If we are given a matrix, how can we tell if the matrix has an inverse?
2. What are some applications of an inverse?
3. Is there a formula for finding the inverse?
4. (What are your thoughts?)

THEOREM 4

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible.

Example:

1) Use the formula above to find the inverse of $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$.

2) Explain why the matrix $B = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ does not have an inverse.

A Theorem that is Actually an Application

Suppose A is an $n \times n$ matrix

$$A\mathbf{x} = \mathbf{b}$$

In general we would have to use ______________________________ and
it is possible that there might be ______________________________ solutions.

However, if A is ______________________________ then the equation $A\mathbf{x} = \mathbf{b}$ will have a _______________ solution.

Proof:

So if A is invertible - what does that mean about A?
How many pivots does it have?
Relate this to 1-1 and onto?

Theorem:

1. Suppose that A is an invertible matrix. Then

$$(A^{-1})^{-1} = A.$$

Proof:

2. **Socks and Shoes**

If A and B are invertible $n \times n$ matrices then so is AB. In fact:

$$(AB)^{-1} = B^{-1}A^{-1}$$

Proof:

Follow Up:

$$(ABC)^{-1} =$$

A Way to Find the Inverse of A

Theorem: An $n \times n$ matrix A has an inverse if and only if it is row equivalent to the identity matrix (i.e. it has n pivots).

Proof: We have already explained why IF A is invertible then A has to have n pivots.

The proof of the other direction follows from the algorithm below:

ALGORITHM FOR FINDING A^{-1}

Row reduce the augmented matrix $[\,A \quad I\,]$. If A is row equivalent to I, then $[\,A \quad I\,]$ is row equivalent to $[\,I \quad A^{-1}\,]$. Otherwise, A does not have an inverse.

EXAMPLE 7 Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$, if it exists.

9. Let $A = \begin{bmatrix} 1 & 2 \\ 5 & 12 \end{bmatrix}$, $\mathbf{b}_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$, and $\mathbf{b}_4 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

a. Find A^{-1}, and use it to solve the four equations $A\mathbf{x} = \mathbf{b}_1$, $A\mathbf{x} = \mathbf{b}_2$, $A\mathbf{x} = \mathbf{b}_3$, $A\mathbf{x} = \mathbf{b}_4$

15 Characterization of Invertible Matrices

Recap:

Last time we saw that sometimes a linear transformation can be "undone". The relationship with matrices involved a new word and we could talk about the ________________________ of a matrix A. We denoted this with a symbol ____________.

We only talk about inverse for square matrices [Got it!]

A matrix is [sometimes/always/never] invertible.

Last time we learned a formula for how to find the inverse of a 2×2 matrix.

We were given an explicit description of this inverse. **This was only for the 2×2 case.**

Query: Have we yet seen any ways to solve the problem below?

EXAMPLE 7 Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$, if it exists.

Teamwork Problem:

1) Suppose that A is an invertible matrix with inverse A^{-1}. What do you think the inverse of A^{-1} is?

Fill in the blanks of the proof below:

"I think the inverse of A^{-1} is ____________. I can verify that my guess is correct, by checking that ____________ satisfies the definition. I will check that

$$(\_\_\_\_\_)A^{-1} = I$$

and also that

$$A^{-1}(\_\_\_\_\_) = I.$$

These are both true because A^{-1} and A are inverse of each other. $\square$"

2) Prove that if A and B are invertible $n \times n$ matrices then the matrix AB is invertible.

Proof: Suppose that A and B are invertible $n \times n$ matrices. Then I claim that the matrix $B^{-1}A^{-1}$ is the inverse of the matrix AB. I will check that this matrix satisfies the definition. I will check that [fill in the blanks, and also explain why the left hand side simplifies down to the identity.]

$$(\_\_\_\_\_)(AB) = I.$$

and also that

$$AB(\_\_\_\_\_) = I.$$

□

3) What do you think the inverse of ABC would be (if A, B, C are all invertible matrices)? Could you prove it?

4) If A, B and C are invertible $n \times n$ matrices, can you solve the following for the matrix X? [Explain your answer by clearly explaining what you are doing at each stage. Each explanation should be something like "multiply both sides by blah on the [left/right]."

$$C^{-1}(A + X)B^{-1} = I$$

A Theorem that is Actually an Application

Suppose A is an $n \times n$ matrix

$$A\mathbf{x} = \mathbf{b}$$

In general we would have to use ______________________________ and
it is possible that there might be ______________________________ solutions.
However, if A is ______________________________ then the equation $A\mathbf{x} = \mathbf{b}$ will have a _______________ solution.
Proof:

So if A is invertible - what does that mean about A?
How many pivots does it have?

Can you relate this to 1-1 and onto properties? Does this make sense that to undo a transformation T, T has to pair up every element in the domain and codomain uniquely?

Theorem: An $n \times n$ matrix A has an inverse if and only if it is row equivalent to the identity matrix (i.e. it has n pivots).

Proof: We have already explained why IF A is invertible then A has to have n pivots.

The proof of the other direction follows from the algorithm below:

ALGORITHM FOR FINDING A^{-1}

Row reduce the augmented matrix $[A \quad I]$. If A is row equivalent to I, then $[A \quad I]$ is row equivalent to $[I \quad A^{-1}]$. Otherwise, A does not have an inverse.

EXAMPLE 7 Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$, if it exists.

9. Let $A = \begin{bmatrix} 1 & 2 \\ 5 & 12 \end{bmatrix}$, $\mathbf{b}_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$, and $\mathbf{b}_4 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

a. Find A^{-1}, and use it to solve the four equations $A\mathbf{x} = \mathbf{b}_1$, $A\mathbf{x} = \mathbf{b}_2$, $A\mathbf{x} = \mathbf{b}_3$, $A\mathbf{x} = \mathbf{b}_4$

THEOREM 8

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.

b. A is row equivalent to the $n \times n$ identity matrix.

c. A has n pivot positions.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

e. The columns of A form a linearly independent set.

f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j. There is an $n \times n$ matrix C such that $CA = I$.

k. There is an $n \times n$ matrix D such that $AD = I$.

l. A^T is an invertible matrix.

16 Practice with Invertible Matrices (and a preview of Determinants)

Reality Check: In general for an $m \times n$ matrix A if $m \neq n$ then the conditions:

(C) A has a pivot in every column

(R) A has a pivot in every row

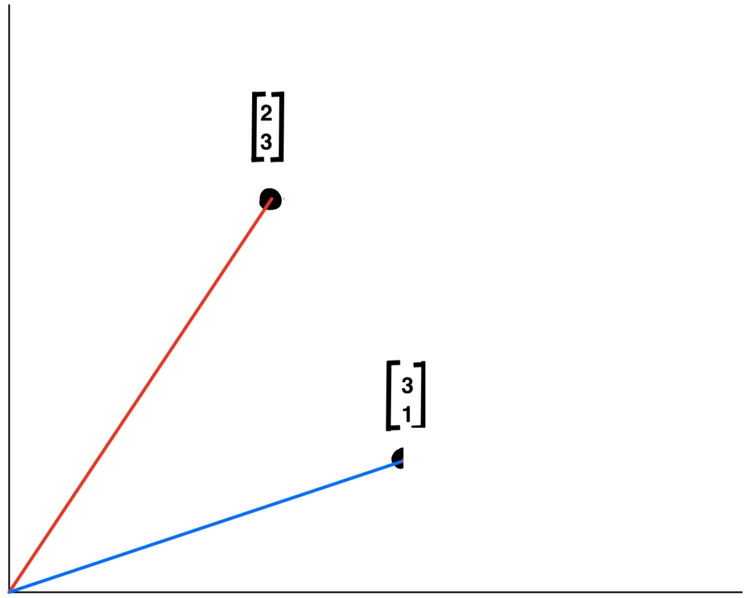
are [different/ the same].

PAUSE: Make sure your group is in agreement about the above answer.

With your group talk about the following vocabulary words and see which of these you would "connect" with the conditions above. Circle the best choice.

$\mathbb{R}^m$ is independent matrix columns baloney	(C) / (R) / Neither / Sentence doesn't even make sense
The columns of A are linearly independent	(C) / (R) / Neither / Sentence doesn't even make sense
The entries of A are all positive numbers	(C) / (R) /Neither /Sentence doesn't even make sense
The columns of A span $\mathbb{R}^m$	(C) / (R) / Neither / Sentence doesn't even make sense
The transformation defined by A is 1-1	(C) / (R) / Neither / Sentence doesn't even make sense
The transformation defined by A is onto	(C) / (R) / Neither / Sentence doesn't even make sense
The system $A\mathbf{x} = \mathbf{b}$ is independent.	(C) / (R) / Neither / Sentence doesn't even make sense
The system $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b} \in \mathbb{R}^m$.	(C) / (R) / Neither / Sentence doesn't even make sense
The system $A\mathbf{x} = \mathbf{b}$ is consistent for some $\mathbf{b} \in \mathbb{R}^m$.	(C) / (R) / Neither / Sentence doesn't even make sense
The matrix is independent	(C) / (R) / Neither / Sentence doesn't even make sense
The system $A\mathbf{x} = \mathbf{0}$ is consistent for some $\mathbf{b} \in \mathbb{R}^m$.	(C) / (R) / Neither / Sentence doesn't even make sense
A the matrix A spans $\mathbb{R}^m$	(C) / (R) / Neither / Sentence doesn't even make sense

Consider the two vectors below. With your group, find some space on one of the white boards and draw these vectors (draw medium-large).



- What are some sentences you can say about these two vectors?
- You can practice writing your sentences on the board.
- If it's helpful, you can call the vectors letters like $\mathbf{u}, \mathbf{v}$.
- Focus on making correct, complete, sentences.
- For each sentence that you make, can you justify it in multiple ways? (Geometric argument? Explanation with vehicles? With pivots?)
- Word bank: span, independent, multiple, combination, matrix, inverse

Reality Check:

For a square matrix $n \times n$, the two conditions:

(C) A has a pivot in every column

(R) A has a pivot in every row

are [different/ the same].

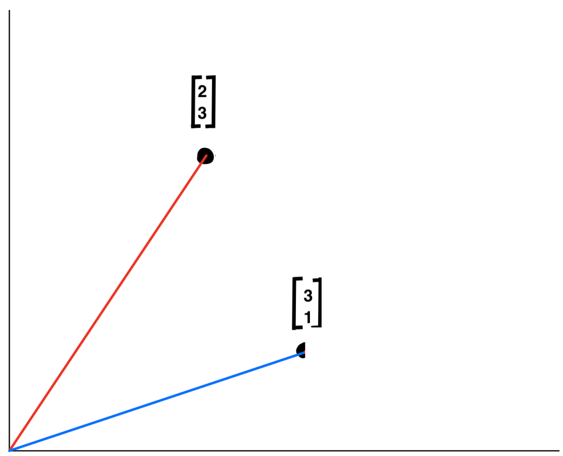
PAUSE: Make sure your group is in agreement about the above answer.

In this all the matrices are $n \times n$.

1. If the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then A is row-equivalent to the $n \times n$ identity matrix. (T/F)
2. If the columns of A span $\mathbb{R}^n$, then the columns are linearly independent (T/F)
3. If A has at least one free variable then A is invertible. (T/F)
4. If A is an $n \times n$ matrix, then the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$. (T/F)
5. If A is an $n \times n$ **invertible** matrix, then the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$. (T/F) (Did you have to go back and change your answer to #4? If so, that's ok - these problems were designed as a pair to help you!)
6. If the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b} \in \mathbb{R}^n$ then the solution is unique for each $\mathbf{b}$. (T/F)
7. If A is an $n \times n$ matrix , then the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for some $\mathbf{b}$ in $\mathbb{R}^n$. (T/F)
8. If A is an $n \times n$ **invertible** matrix, then the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for some $\mathbf{b}$ in $\mathbb{R}^n$. (T/F)

Below you are given some conditions on matrices (or possibly transformations). For each of these discuss with your group whether or not the matrix (or transformation) would be invertible. Your answer for each part should be either [Yes / No / Can't Tell] and you should be able to support your answer with an explanation.

1. A is a 4×4 matrix whose fourth column is equal to the sum of the first three columns.
2. B is a 7×7 matrix and the equation $B\mathbf{x} = \mathbf{b}$ has a solution for **some** vector $\mathbf{b} \in \mathbb{R}^7$.
3. C is the matrix $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$
4. G is a 5×5 matrix such that the equation $G\mathbf{x} = \mathbf{y}$ has more than one solution for **some** $\mathbf{y} \in \mathbb{R}^5$.
5. H is a 4×4 matrix with two identical columns.
6. I is the identity matrix.
7. J is a 7×7 matrix whose columns do not span $\mathbb{R}^7$.
8. $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation that has the property that $T(\mathbf{u}) = T(\mathbf{v})$ for two different vectors $\mathbf{u}, \mathbf{v}$.
9. K is the matrix ABC where A, B, C are invertible 5×5 matrices.
10. L is the matrix corresponding to the operation "rotate counterclockwise by 20 degrees"
11. M is the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
12. N is a matrix that has a row of all zeros.
13. O is the 3×3 matrix of all zeros.
14. P is a matrix whose columns are the two labeled vectors in the picture below.



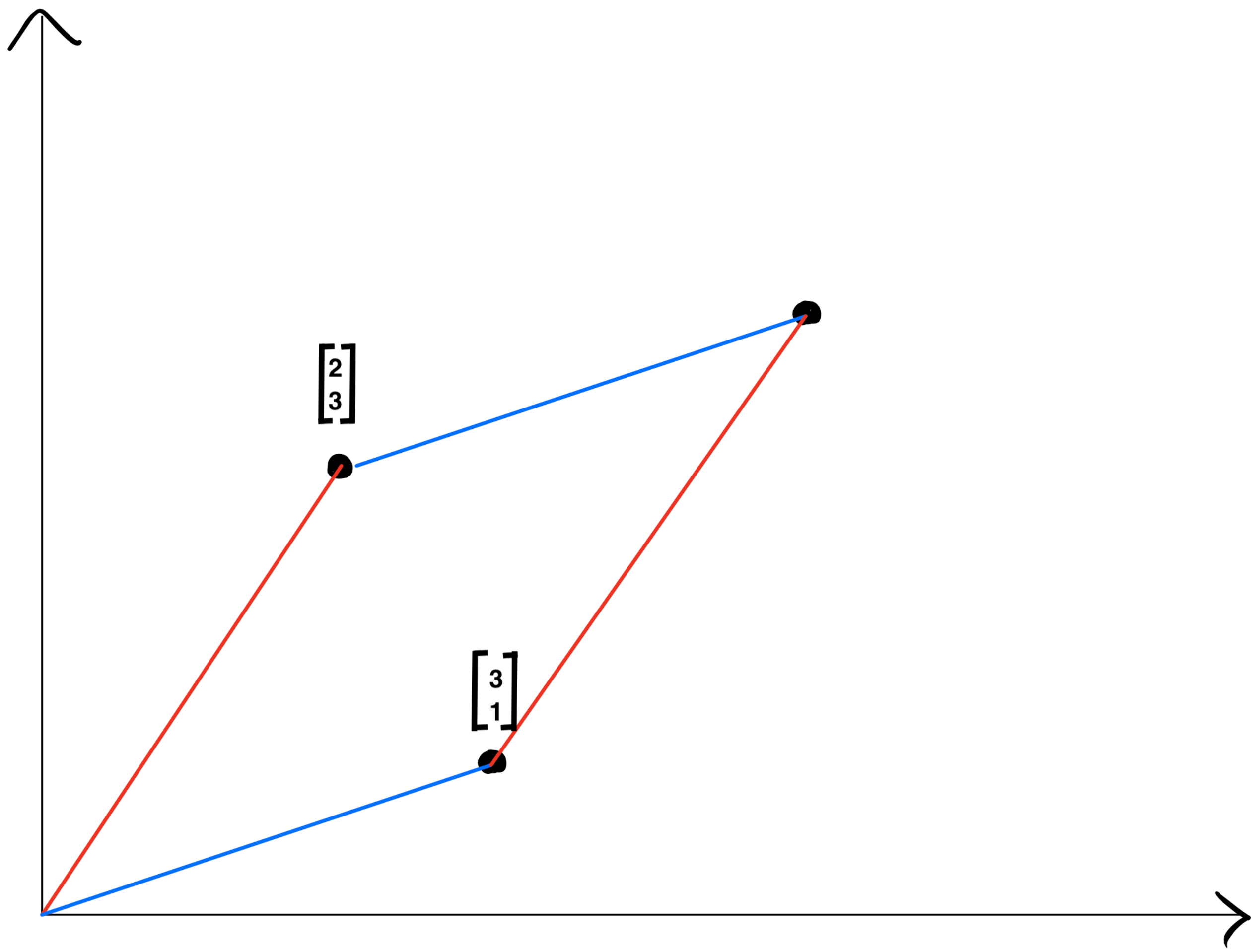
Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation given by

$$T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -9x_1 + 7x_2 \\ 4x_1 - 3x_2 \end{bmatrix}.$$

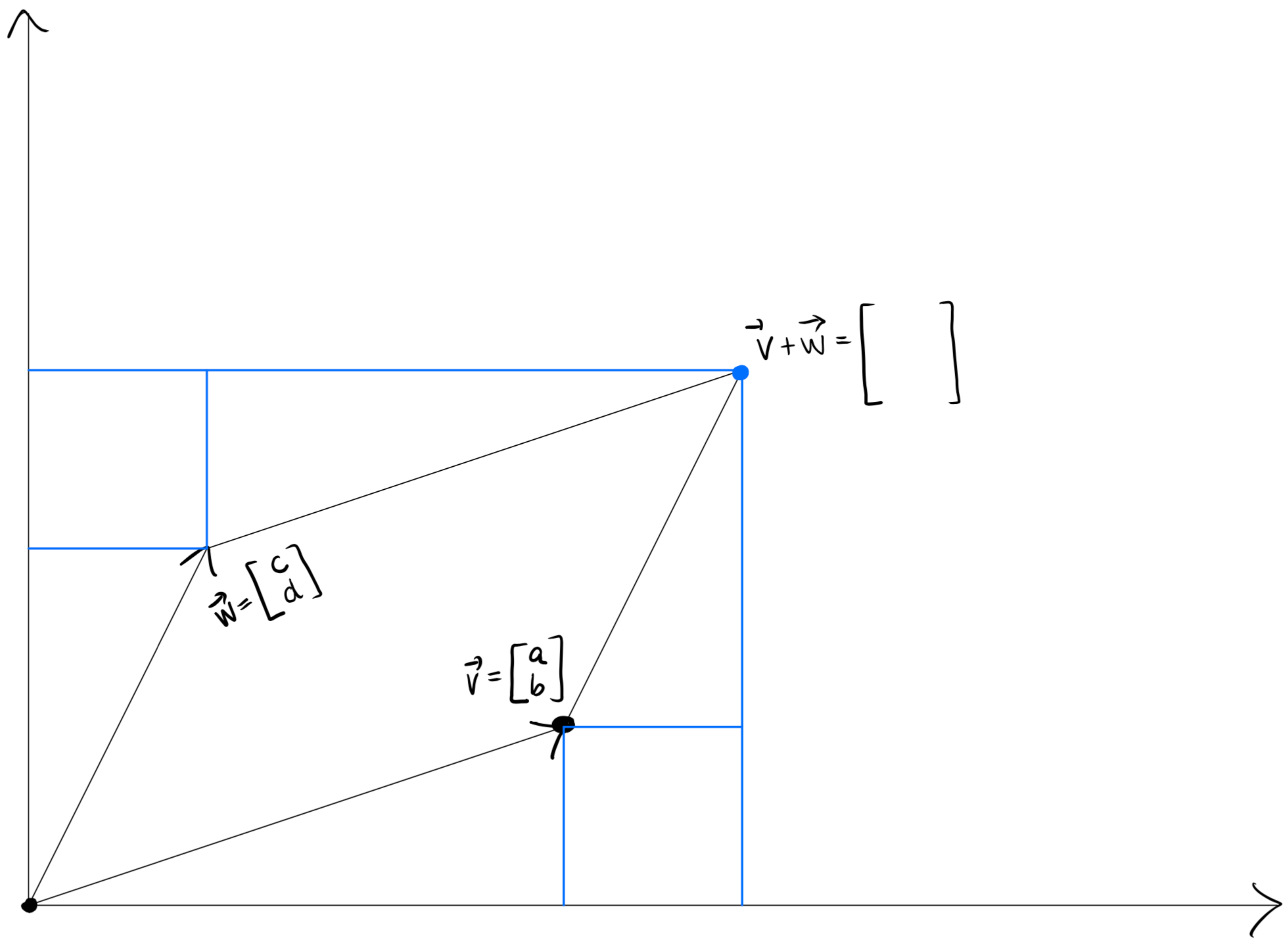
Determine if T is invertible and find the inverse of T. (Hint: convert this to something about matrices)

Old Man Gauss is building a fence. (Why not?) He wants his fenced in region to be as in the picture below.

1. What is the area of the region the Old Man will fence in?



2. Now try the same with the picture below. Do you notice any connection between your answer and something we have seen in class?



17 Introduction to Determinants

Some Review for the Exam:

- The exam will be very similar to questions you have seen before.
- I will put lots of review questions on the online HW for Wednesday (I will extend this HW until Friday)
- In terms of computation, I might ask you to set-up (but not solve) and explain how you would use a computer to help you solve a particular question: Example: Here are 3 vectors in $\mathbb{R}^4$ what would you do to determine if one of those vectors could be written as a linear combination of the other? Example: What would the augmented matrix be for the equation $A\mathbf{x} = 2\mathbf{x}$ (Hint: think about the Quiz4 you'll get back on Wednesday.)
- Be sure to give complete explanations when you are writing. For instance, it will not be ok to say "pivot in every column." Instead you want to say something like: "To solve this, I would form the following augmented matrix and I would look for its RREF. If there is not a pivot in the last column, I would conclude X and if there were, I would conclude ..."
- I might ask you to multiply matrices. Please make sure you can do this confidently and reasonably quickly. I will not be able to allow additional time on the exam.
- Most of the exam will feel very conceptual (as most of the course has felt). Suppose you are given a 4×7 matrix. Is it possible for the columns to be linearly independent? Why or why not? If this represented the augmented matrix for a system of equations, could this system be consistent?
- You should be familiar with the proofs from our last two homework assignments. I am unlikely to ask you to write long technical proofs, but I might ask you to supply examples or write short proofs: "Prove that there are two matrices A and B such that A and B are invertible by $A - B$ is NOT invertible. Justify your answer." "Solve for X in the expression $B(I + X)C = D$, assuming B and C are invertible matrices." "Prove that the transformation defined by BLAH is NOT linear by explicitly showing why the linearity properties fail."

Warmup:

Consider the following two statements about an $m \times n$ matrix A and a vector $\mathbf{b} \in \mathbb{R}^m$:

Statement 1) The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for every $\mathbf{b}$ in $\mathbb{R}^m$.

Statement 2) The equation $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $\mathbf{b}$ in $\mathbb{R}^m$.

Discussion Questions:

Suppose you know that Statement 1 is true. Does this sometimes/always/never mean that Statement 2 is true?

Suppose you know that Statement 2 is true. Does this sometimes/always/never mean that Statement 1 is true?

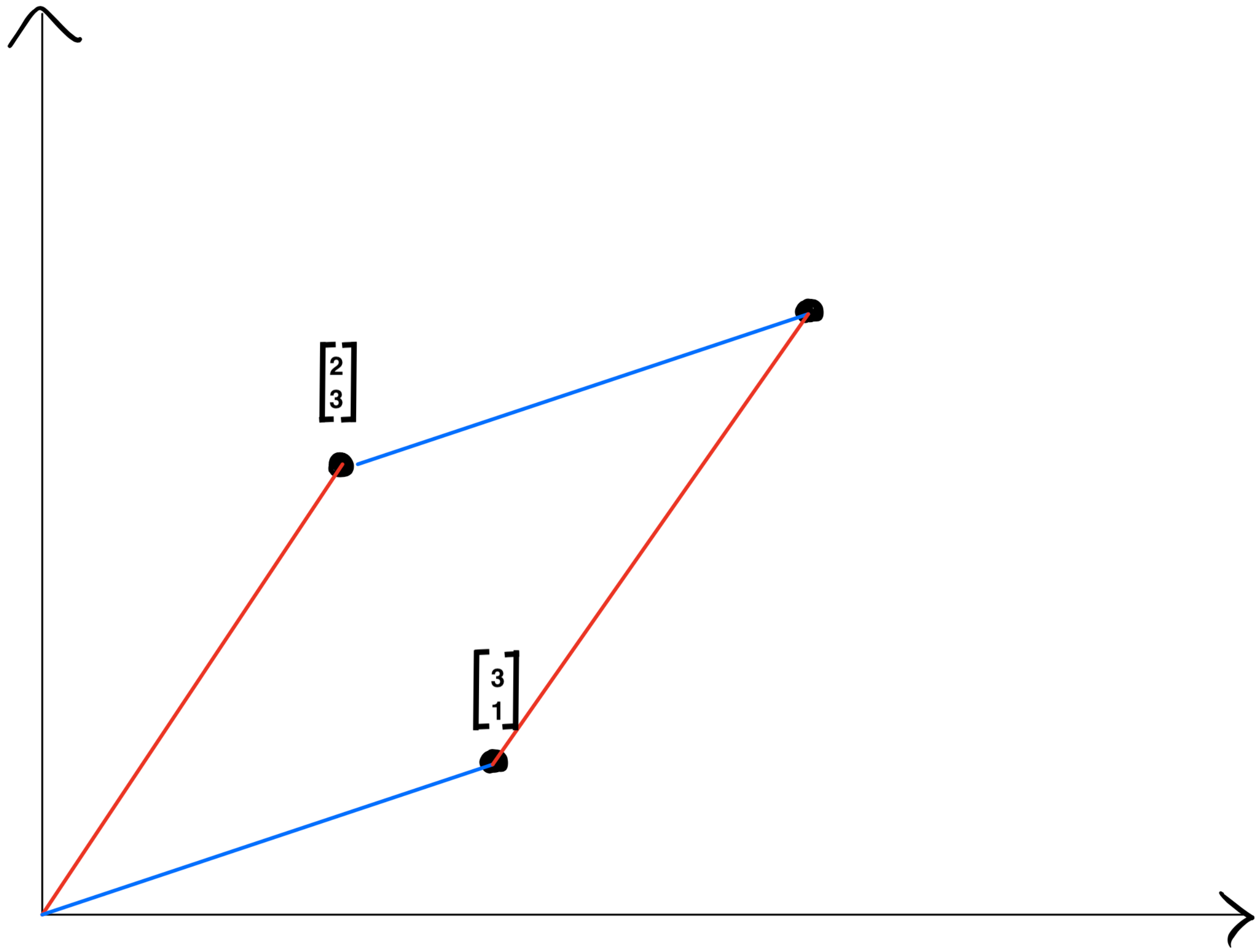
What if in addition you knew that $m = n$. Would your answers above change? Why or why not?

Warmup 2:

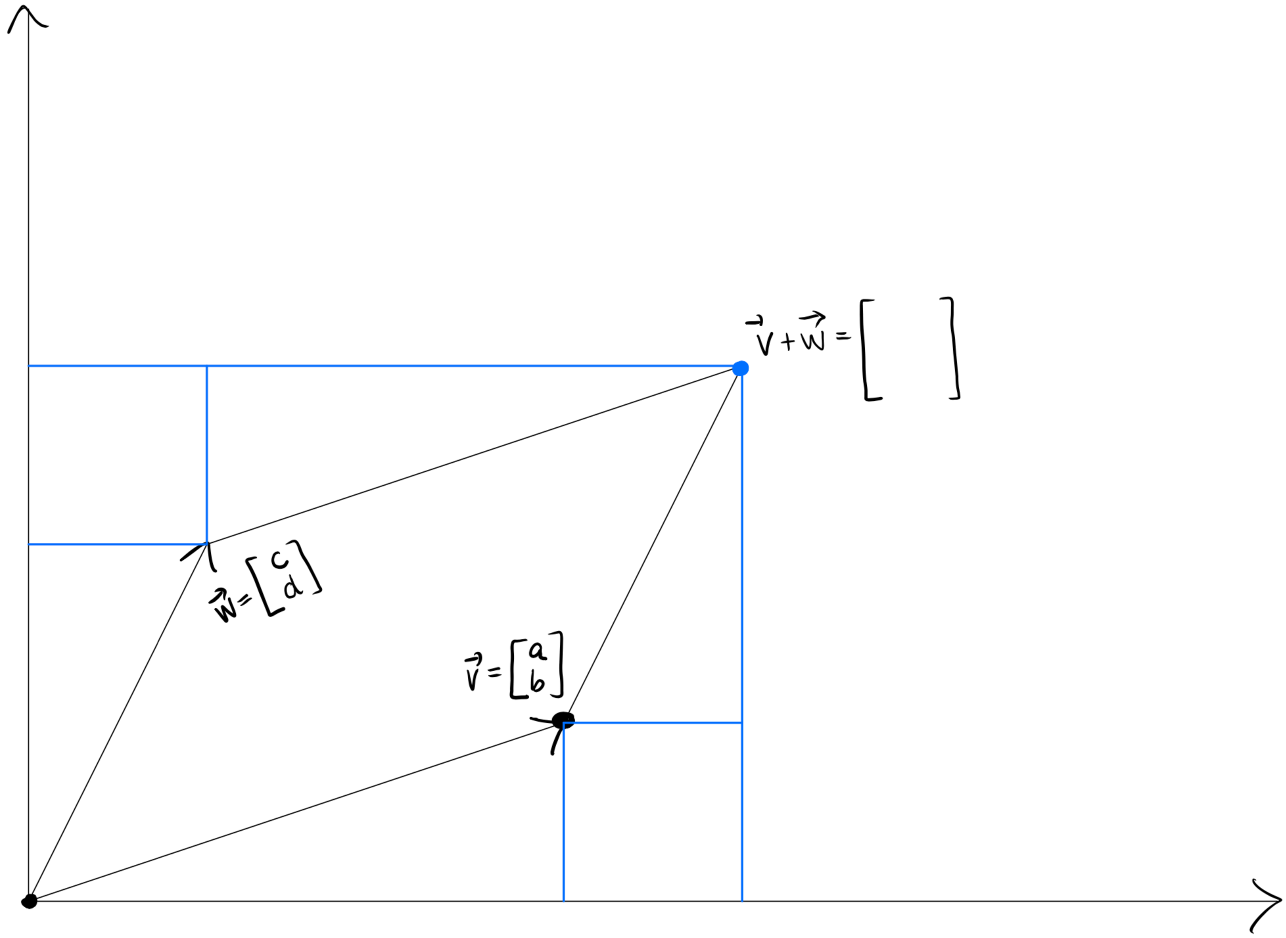
Find the inverse of the matrix $\begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix}$ using the formula that is written on the board.

Old Man Gauss is building a fence. (Why not?) He wants his fenced in region to be as in the picture below.

1. What is the area of the region the Old Man will fence in?



2. Now try the same with the picture below. Do you notice any connection between your answer and something we have seen in class?



We have just seen that the area spanned by the vectors $\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}$ is equal to $ad - bc$. Or have we....

Reality Check:

Find the area of the region spanned by the columns of the matrix: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Important Concept and Definition

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a 2×2 matrix. Then the number $ad - bc$ is called the **determinant** of the matrix, denoted $\det A$.

Some properties:

- $\det A \neq 0$ if and only if A is invertible.
- $|\det A|$ is the area of the **parallelogram** spanned by A.
- It is useful to think about $\det A$ as measuring the ______________ by which the **area** of the unit square has changed after the linear transformation due to A.

Example:

In thinking about the following examples (WITHOUT thinking about any matrices) what would the following transformations do to the **area** of the unit square?

1. Rotation by 45 degrees clockwise
2. Stretching by a factor of 3 in the x direction and 2 in the y direction
3. Reflection across the x axis.
4. The transformation obtained by:

   1) Flip a coin. If it's heads, reflect across the y-axis

   2) If it's tails, use a random number generator to get a random number between 0 and 360 degrees. Rotate the plane clockwise by that angle.

   3) Repeat this process 2022 times.
5. Projection onto the y-axis.

Example: Suppose that $A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 2 \\ 1 & 2 \end{bmatrix}$

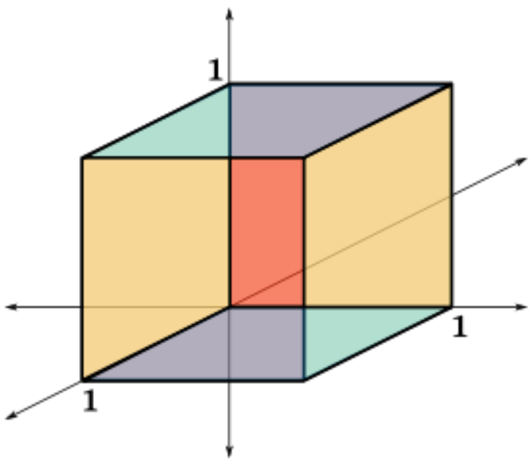
Calculate the following:

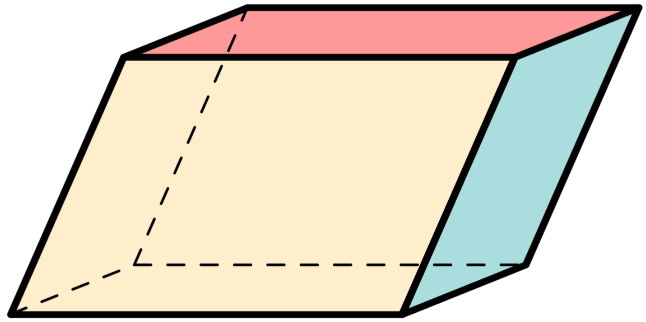
$$AB, BA, \det(A), \det(B), \det(AB), \det(BA)$$

Can you think of any reason that explains the relationship you just found?

A Fundamental Property of Determinants

What happens in higher dimensions?





In general the determinant of an 3×3 measures change in n-dimension volume after we apply a transformation. (Up to a sign.)

DEFINITION For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. In symbols,

$$\begin{aligned}\det A &= a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}\end{aligned}$$

EXAMPLE 1 Compute the determinant of

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

20 More Practice With Determinants

Important properties of Determinants

- To each $n \times n$ matrix, we associate a number $\det(A)$ called the determinant of A.
- The ________________________ of this number measures the factor by which the transformation determined by A changes n−dimensional **volume.**
- There is a way to calculate determinants by expanding across rows / columns.

Example: Find the determinant of the matrix below

$$\begin{vmatrix} 6 & 0 & 2 & 4 & 0 \\ 9 & 0 & -4 & 1 & 0 \\ 8 & -5 & 6 & 7 & 1 \\ 2 & 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 & 0 \end{vmatrix}$$

What would the determinant of the matrix U be?

$$U = \begin{bmatrix} 2 & 4 & 5 & 6 \\ 0 & -1 & 5 & 13 \\ 0 & 0 & 5 & 1095 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

This matrix is called ________________________ and

Theorem: The determinant of an upper triangular matrix is equal to the

Since we can use row operations to put a matrix into Echelon Form - do those change the determinant?

Quick reality check: Is the matrix $\begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$ row equivalent to the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$? Do they have the same determinant?

Let's see how: how do Row Operations Affect Determinants?

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

- Switching two rows will ______

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a & b \\ kc & kd \end{bmatrix}$

- Replacing R_i with kR_i will ______

$\begin{bmatrix} 6 & 5 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 6 & 5 \\ 3+6k & 4+5k \end{bmatrix}$

- Replacing R_i with $R_i + kR_j$ will ______

More complex examples:

Example: • Replacing R_i with $2R_i - 4R_j$ will ______

Example: • Replacing R_i with $-3R_i - 4R_j$ will ______

Example Suppose that a student takes a matrix B and performs the following operations:

- Replaces row 2 with $3(R_1) - 4R_2$
- Divides row 3 by 2
- Replaces row 4 by $(-5)(R_1) + R_4$
- Switches rows 4 and 3
- Replaces row 2 with $3R_2 - 2R_3$

and obtains the matrix: $\begin{bmatrix} 1 & 0 & 10 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$. Calculate the determinant of the original matrix.

A segue into new material:

So far every **vector** we have worked with ALWAYS an element of $\mathbb{R}^m$ for some m. What was important about vectors? (Let's list some properties) Are there any other "things" that we can do similar things to?

21 Vector Spaces

Important Definition

A ________________________ is a nonempty set V of objects, called ____________,

on which are defined two operations, called ____________ and ____________, subject to the ten axioms

(or rules) listed below. The axioms must hold for all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V and for all scalars c and d.

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} + \mathbf{v}$ is ____________
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. There is a special vector called the ________________________ in V with the property that
5. For each $\mathbf{u}$ in V there is a vector $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) =$ ____________
6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$ is ____________
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$
10. $1\mathbf{u} = \mathbf{u}$

The MAIN things you should concern yourself with are:
Ok I have a set V of "things" aka "vectors"

- When I add two things in the set, do I get an thing in that set?
- When I multiply a thing by a scalar, do I get another thing in that set?
- Is 0 in the set?

Example: The set $\mathbb{R}^3$ of all vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is a **vector space**.

So is the set $\mathbb{R}^7$.

$\mathbb{R}^n$ is a vector space for all $n \geq 1$.

So far - this is the **main example** of vector space we have seen.

Although so far in our class we have focused on studying "vectors" as being synonymous with "elements of $\mathbb{R}^n$" we are now going to extend our study to more general vector spaces.

Why?

- We have studied pretty extensively, the properties of linear independence, span, linear transformations, and how these properties interact with each other.
- Maybe we can use all this hard work to understand things outside of just "column vectors like $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$"
- For instance, we can justify statements like: "If $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation then the standard matrix for T would be ____________________ and thus it cannot have a pivot in every [row/column]. Thus the transformation T is not __________. So this means that there must be two __________ inputs that go to the __________ output."
- We are going to see that Linear Algebra (with only a tiny bit more work than what we've already done) can allow us to generalize this to something like: "Suppose $T : V_1 \to V_2$ is any **linear transformation** from one vector space to another. Then if the **dimension** of V_2 is less than the dimension of V_1 then this must mean that T will send different inputs to the same output. In other words, the transformation T is not **lossless**.
- Here's another example: "Any set of 3 vectors in $\mathbb{R}^2$ must be linearly dependent. That means that one can be written in terms of the other."
- We will generalize this to: "Any set of 3 vectors in a 2-dimensional vector space V form a linearly dependent set. So that means that one can be written as a combination of the other."

We are going to learn about all of this stuff over the next month or so. For now, let's just see some examples.

Example:

Let $\mathbb{P}$ be the set of all polynomials $p(t)$ with real coefficients. For example, V contains things like

$$2t + t^2$$

and

$$t^{100} - 3.14t^2 - 16.$$

So in this example "vectors" are polynomials. Is V a vector space?

- Ok, so if V is a vector space, then e.g. $t, t+1$ and $t+2$ all all "vectors." Here is an example of a linear combination of these three polynomials:

  $$4(t) + 2(t+1) - 3(t+2) = 3t - 4.$$

  Talk in your group about what the coefficients of this linear combination were. Talk about why this means that

  "The vector ____________________________ is in the **span** of the vectors $t, t+1, t+2$."

  As before Linear combinations are just ways of combining vectors with coefficients!

- Recall that a set of vectors, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is called **linearly independent** if the only solution to

  $$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

  is if ALL the coefficients c_1, c_2, c_3 are zero. In other words, the only way to get 0 as a linear combination of the vectors v, is by using all zero coefficients.

- Determine whether or not $\{t, t+1, t+2\}$ is a linearly independent set of **vectors** in $\mathbb{P}$.

Example: The set V of all functions $\mathbf{f} : \mathbb{R} \to \mathbb{R}$ is a vector space if we use the usual definition of + and scalar multiplication.

Let's talk through some of the axioms.

- What do you think is the zero-vector?
- If $\mathbf{f} = \sin x$ what is $-\mathbf{f}$?

What would a linear combination of the **vectors**: $x, \sin x, e^x$ look like?

Do you think the set of vectors: $\{\sin^2 x, \cos^2 x, 5\}$ is linearly dependent?

(Non)-Example: Now let's do an example that is NOT a vector space. Let V be the set of all polynomials of degree 2 and the 0 polynomial.

For instance V contains things like: $0, \quad t^2 + 2t - 5, \quad 3t^2 - 5t$

but does NOT contain things like $3t - 1,\ t^3$.

Explain why V is *not* a vector space. Which of the properties does it violate: (Hint: Look at the 3 MAIN properties on the first page, these should be the first ones you check when looking for vector spaces)

Are you stuck? Are you lost in what we're doing - let's get grounded again by writing down 5 different elements of V: Then we're going to think about the MAIN properties.

Your turn

Which of the following are vector spaces? Remember to check the MAIN properties. Some are **vector spaces** and others are not.

Let U_1 be the set of all vectors in $\mathbb{R}^2$ of the form $\begin{bmatrix} a \\ b \end{bmatrix}$ where $a, b \geq 0$. Draw a picture of U_1 as a **subset of** $\mathbb{R}^2$**.**

Let U_2 be the set of all vectors in $\mathbb{R}^2$ of the form $\begin{bmatrix} a \\ b \end{bmatrix}$ where $ab \geq 0$. Draw a picture of U_2 as a **subset of** $\mathbb{R}^2$. Hint: What is this saying about the signs of the numbers a, b? Use this to draw your picture.

Let U_3 be the set of all vectors in $\mathbb{R}^2$ of the form $\begin{bmatrix} a \\ b \end{bmatrix}$ where $b = 4a$. Draw a picture of U_3 as a **subset** of $\mathbb{R}^2$.

Let U_4 be the set of all vectors in $\mathbb{R}^2$ of the form $\begin{bmatrix} a \\ b \end{bmatrix}$ where $b + a = 2$. Draw a picture of U_4 as a **subset** of $\mathbb{R}^2$.

What is a Subspace?

DEFINITION

> A **subspace** of a vector space V is a subset H of V that has three properties:
>
> a. The zero vector of V is in H.[2]
>
> b. H is closed under vector addition. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u}+\mathbf{v}$ is in H.
>
> c. H is closed under multiplication by scalars. That is, for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

(Together as a class) We have previously talked a lot about what it means to be a **subspace of $\mathbb{R}^n$.** This is where the big vector space was $V = \mathbb{R}^n$.

For instance, let

$$H = \{\begin{bmatrix} a \\ b \end{bmatrix} \mid a^2 + b^2 \leq 1.\}$$

Is H a **subset** of $\mathbb{R}^2$?

Is H a **subspace** of $\mathbb{R}^2$?

Main Idea: A "subspace of V" is a "smaller (though could be equal)" vector space that lives inside of V.

Note: Subspace is NOT the same as subset. It means subset AND the three properties in the box above.

Every subspace of V is a subset, but not every subset will be a subspace.

Question: Which of the subsets of $\mathbb{R}^2$ on the previous page are **subspaces** of $\mathbb{R}^2$?

Let V be the set of all functions. This is our BIG vector space. Now inside of V let's consider all differentiable functions that satisfy the following equation:

$$W = \{f(x) \in V \ : \ f'(x) = f(x)\}.$$

Question: Does this set form a **subspace** of V?

We need to check three things:

Let

$$W_2 = \{f(x) \in V \ : \ f''(x) = 0\}.$$

Can you find a set of "vectors" that span the set W_2?

Let

$$P_n = \{ \text{ all polynomials of degree} \leq n \ \}$$

.

For instance P_3 contains polynomials like $t^3 + t^2$ and $t^2 - t$.

- Is P_n a vector space? (Answer: Yes - the main point is to check that the sum or scalar multiple of vectors of degree $\leq n$ still has degree $\leq n$.)
- Let's work with
$$P_2 = \{ \text{ all polynomials of degree} \leq 2 \ \}.$$
- Do the polynomials $\{1, t, t^2\}$ **span** all of P_2? (i.e. can every polynomial be written as a linear combination of these three?)
- Do the vectors in $\{1, t, t^2, t^2 + t\}$ **span** all of P_2?
- Explain why the vectors in $\{t^2, t^2 + 3t - t, t^2 + t, t^2 + 5t\}$ do NOT span all of P_2?
- Is the polynomial t^2 in the span of $\{1 - t, t - t^2, 1 - t^2\}$? Write down an equation (with coefficients c_1, c_2, c_3 that you could use to check this. You might not have time to solve this equation, but that's ok, what's important is practicing writing down the relevant equation.

Let

$$H = \{a + bt^2 \ : \ a, b \in \mathbb{R}\}.$$

is H a **subset** of P_2? (Look back for the definition of P_2 if you need to). Write down three or 4 different elements in H.

Is H a **subspace** of P_2?

Is $H = P_2$?

Can you find a set of vectors that span H?

If a set of vectors in H **spans** H then we might say that these vectors **generate** H.

Which of the following sets will span H? Are the vectors linearly independent?

$\{1, t^2\}$

$\{1, t, t^2\}$

$\{1, 1 + t^2\}$

$\{1 + t^2, 2 + 2t^2\}$

$\{1 + t^2, 1 - t^2\}$

$\{1 + 2t^2\}$

Do you notice anything? We'll be exploring this in more detail soon!

Let

$$H = \{a + t^2\}$$

Is H a **subset** of P_1? (Look back for the definition of P_2 if you need to).

is H a **subset** of P_2? (Look back for the definition of P_2 if you need to).

Is H a **subspace** of P_2?

22 Subspaces, Proofs

We saw that a subspace H of a vector space V is a ____________________ with the **additional** properties that

- The zero vector is in H.
- Closed under addition: If two things are in H then so is their sum.
- Closed under scalar multiplication: If a thing is in H then so are all scalar multiples of it.

Warmup: Which of the following are subspaces of $\mathbb{R}^2$. If they are NOT, find a violation of the subspace definition.

1. H is the union of the x and y axes, $H = \{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1 = 0 \text{ or } x_2 = 0.\}$

2. $H = \{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1 + x_2 = 5\}$

3. $H = \{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1 + x_2 = 0\}$

4. $H = \{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1^2 + x_2^2 = 25\}$.

Which of the following are subspaces of the vector space V of all functions $\mathbf{f} : \mathbb{R} \to \mathbb{R}$. Here the zero vector is the function $f(x) = 0$.

1. $H = \{f \in V : \; f(0) = 5.\}$

2. $K = \{a \sin x + b \cos x : a, b \in \mathbb{R}\}$

3. $L = \{f \in V : f''(x) = 0\}$

Are you stuck? Did you remember to check the zero vector? Can you write down some typical elements from your set?

Did you notice?

In the previous examples that **were** subspaces, could you write them as the span of a set of vectors?

Theorem: If V is a vector space and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are vectors in V, then $\text{Span}\{\mathbf{v_1}, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is a subspace of V.

Proof:

Bonus:

It is actually true that every subspace is the span of a set of vectors. In fact, if you choose the "minimal" number of such vectors, then this will help us determine the **dimension** of this space.

Example: Show that the set H defined below is a subspace of $\mathbb{R}^4$:

$$H = \left\{ \begin{bmatrix} a - 3b \\ b - a \\ a \\ b \end{bmatrix} \ : \ a, b, \in \mathbb{R} \right\}.$$

Proof: We will use the previous ____________ and show that H is equal to ________________________.

.

That worked well. It won't always be so easy to write something as a Span. But it doesn't hurt to try.

Here's a harder example. Consider the set P_3 of all polynomials of degree ≤ 3, that is, all polynomials of the form

$$at^3 + bt^2 + ct + d.$$

We will define a subset $H = \{\text{polynomials } p(t) \text{ such that } p(2) = p(3)\}$.

Step 1: That the heck is going on? Let's check that the following polynomials are in H:

$$p(t) = 0, \quad q(t) = (t-2)(t-3), \quad r(t) = t^2 - 5t, \quad s(t) = 1.$$

There are infinitely many polynomials in H!

Step 2: How could we decide if we think H is a **subspace** of P_3.

Zero vector?

If we add two things in H, will their sum be in H? What about scalar multiple?

Q: If $p(2) = p(3)$ and $q(2) = q(3)$ then will $(p+q)(2) = (p+q)(3)$?

Q: If $p(2) = p(3)$ and $c \in \mathbb{R}$ then will $(cp)(2) = (cp)(3)$?

It turns out that $H = \text{Span}\{t^3 - 19t, t^2 - 5t, 1\}$.

23 Enter Transformations: Column Space, Null Space, Kernel and Range

Warmup:

Here is a new definition, but I bet you can guess what it will say!

Important Definition

Suppose that V and W are vector spaces. A function $T : V \to W$ is called a **linear transformation** if

Examples:

We will still work with P_n. Define a function by the following rule.

$$T(p(t)) = \frac{d}{dt}\, p(t)$$

Reality Check : What is $T(t^3 + 3t)$?

Do you think $T((t^3 + 3t) \quad + \quad (t^2 + t))$ will be the same as $T(t^3 + 3t) + T(t^2 + t)$?

Do you think $T(5(t^3 + 3t))$ will be the same as $5T(t^3 + 3t)$?

Question: Is this function linear?

$T(p(t) + q(t)) =$

$T(c \cdot p(t)) =$

Which of the following would be a possible definition for the domain and codomain of T? (Hint: Three of these have answer YES and 1 has answer NO)

$T : P_2 \to P_2$

$T : P_4 \to P_2$

$T : P_2 \to P_3$

$T : P_3 \to P_2$

Let's focus on the linear transformation: $T : P_2 \to P_3$ given by $T(p(t)) = p'(t)$. Let's think about the picture below.

Questions:

- What is the **range** of T?
- Is it all of P_3?
- Is it an onto function?
- How could you describe the range of T?
- Do you think the range of T is a **subspace** of P_3?

Theorem:
If V and W are vector spaces and $T : V \to W$ is a linear transformation then

Before we do the proof we need to prove a

Lemma: If V and W are vector spaces and $T : V \to W$ is a linear transformation then

Proof of Lemma:

Proof of Theorem

Example:

Let $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 0 & -1 & 5 & 4 \end{bmatrix}$. This matrix will define a linear transformation.

Draw a picture of the domain and codomain. List some inputs and outputs.

The range of T can be described as the ____.

For this reason it is called the ____.

Example: The column space of a 5×3 matrix will be a subspace of ______________.

Let's consider the linear transformation $T : P_3 \to P_2$ given by $T(p(t)) = p''(t)$. (Note we are taking two derivatives.) Let's think about the picture below.

Important Definition

Let V and W be vector spaces and let $T : V \to W$ be a linear transformation. Then

Theorem:

Let V and W be vector spaces and let $T : V \to W$ be a linear transformation. Then

Proof:

Let $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 0 & -1 & 5 & 4 \end{bmatrix}$. This matrix will define a linear transformation

What is the **kernel** of T? Can we solve for which vectors are in the kernel of T?

This method will always yield the **kernel** of a matrix transformation as the span of a set of vectors.

25 Review Day + Group Quiz

26 What is a Basis?

Example 1 Let's think about the following situation in $\mathbb{R}^3$ (after all, we live in a 3D world, right?)

You have three vehicles, aptly named $\mathbf{u}, \mathbf{v}, \mathbf{w}$. As always, you can combine the vehicles, going forwards backwards etc.

This means the places you can go is equal to ____________.

Here are the vectors you find in your garage:

$$\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -2 \\ -1 \\ -3 \end{bmatrix}.$$

Describe the places that you can get with these three vectors:

$$\text{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} = [\text{a line in } \mathbb{R}^3 \text{ / a plane in } \mathbb{R}^3 \text{ / all of } \mathbb{R}^3]$$

It's time to let go of things that do not spark joy. Do we really need all three of these vehicles? Which ones do we need? Can we get rid of all of them?

Example 2:

Your twin in the universe P_3 goes out to their garage and find the following five vehicles:

$$1, t, t+1, t^2, t^2+t$$

Like before, the set of all places you can go is equal to _________________________.

Let's get rid of some vehicles: Are there are any that are not necessary?

Can our twin get to every part of P_3? Are there any places they cannot get to?

Do these vehicles span all of P_3? [Yes all of it / No, only part of it].

Is there a different set of vehicles that would still span exactly the same thing as before?

Important Definition

Let H be a subspace of a vector space V.
We say that a set of set of vectors $\mathcal{B}$ is a __________________________ if:

1.

2.

A "basis for H" is a "minimal system of vehicles that span everything".

Example: Let's write down some different **bases** (the plural of basis) for $\mathbb{R}^2$.

We just saw that a vector space (or subspace) can have __________________________

Example: In $\mathbb{R}^n$, the vectors $\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}$ form a **basis** for $\mathbb{R}^n$. This is called the "standard basis" for $\mathbb{R}^n$.

Example: Are the following vectors a basis for $\mathbb{R}^3$? How would we tell?

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}, \text{ and } \mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}.$$

Example: Is the set $B = \{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}0\\1\\3\end{bmatrix}\}$ a basis for $\mathbb{R}^3$?

Is B a basis for $\mathbb{R}^2$? [No! These are vectors in $\mathbb{R}^3$]

B IS a basis for ____.

Example: Find a basis for the vector space P_3

Example: You look at the matrix $A = \begin{bmatrix}1 & 0\\2 & 1\\0 & 3\end{bmatrix}$. The column space of A, which is the span of the columns of A is a subspace of ____________.

A basis for Col(A) is

Important Facts: Let H be a subspace of a vector space V. If a set B of vectors spans H then there are two cases.

1. If B is an independent set, then B is a basis for H
2. If B is a dependent set, then this this means one vector can be written in terms of the others. That vector can be removed from B and the new (smaller) set will still span the same as it did before. We have removed a redundant vector.

This describes an algorithm for constructing a **basis** as long as you have a set of vectors that spans your space.

Null Space and Column Space:

- Is there a way to write down a **basis** for the set Null Space and/or Column Space for a matrix A?
- What are applications of bases? Why are they so useful?

Let $A = \begin{bmatrix} 0 & 2 & 6 \\ 2 & 2 & 16 \\ -1 & 0 & -5 \end{bmatrix}$. Its RREF is $\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

Warmup:

Use this to solve the system $A\mathbf{x} = \mathbf{0}$.

Solutions to this equation form the ______________ of this matrix.

This procedure will always produce a **basis** for the null space of a matrix.

Now let's move on to the **column space** of A.

What if the situation is more complicated?

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix} \sim B = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Algorithm for finding the Column Space:

Start with an original matrix A. A basis for the column space will be

"the columns of A that are pivot columns."

Warning: You need to take the columns of the original matrix. If you take columns from the RREF matrix, this may not be correct.

27 What can we do with a basis?

Warmup:

Here is a matrix and its RREF echelon form:

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix} \sim B = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

What is a basis for Col(A)?

Warning:

Be careful, when finding the column space of a matrix, you always need to take the pivot columns of the **original matrix.**

List one reason why the pivot columns of B can't possibly be a basis for Col(A).

Definition:
If V is a vector space (or a subspace) and has a basis with n elements, then we will say that V is

(Note: We will later prove that this is well-defined, that any two bases will always have the same number of elements.)

Example:

$\mathbb{R}^4$

Col(A) in the warmup is a

Span$\{1, t, t^2, t+1\}$ is

Bases are so interesting, because they allow us to talk about ____________, in a ____________ way.

Theorem:

Let V be a vector space. Let ____________ be a ____________

and suppose that $\mathcal{B}$ is ________________________. Then

Proof sketch: Let $\mathbf{v}$ be vector in H and let's say that $\mathcal{B} = \{\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_n}\}$.

(Why do we know that $\mathbf{v}$ can be written as a linear combination of the elements of $\mathcal{B}$? What property of **basis** are we using?)

Now why do we know that $\mathbf{v}$ can be written in a unique way? Let's argue by contradiction: Suppose that $\mathbf{v}$ can be written in two different ways:

Important

Process: If we have a basis $\mathcal{B} = \{\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_n}\}$ and we write a vector $\mathbf{x}$ in terms of the $\mathbf{b}_i$ as

$$\mathbf{x} =$$

then we call these numbers $c_1, \ldots, c_n$ the ___________________________

If $c_1, \ldots, c_n$ are the $\mathcal{B}$-coordinates of $\mathbf{x}$, then the vector in $\mathbb{R}^n$

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is called the **coordinate vector of x (relative to $\mathcal{B}$)**.

Example:

You are working in P_2 and are given the basis $\mathcal{B} = \{2, 1+t, t^2\}$.

1) Find the $\mathcal{B}$ coordinates for the following vectors

$t^2 + 2t$

t

1

2) What vector is $\mathbf{x}$ if you are given that $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$?

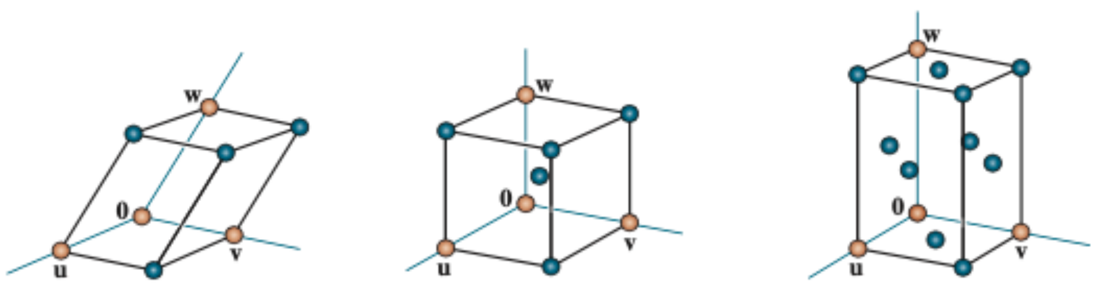
Example:

Suppose you have the following basis for $\mathbb{R}^2$: $\mathcal{B} = \{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-2\\2\end{bmatrix}\}$

1) What vector is $\mathbf{x}$ if $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}-2\\3\end{bmatrix}$?

2) Find the coordinates of $\mathbf{x} = \begin{bmatrix}0\\6\end{bmatrix}$.

Example:
If $\mathcal{E} = \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ is the standard basis of $\mathbb{R}^3$. Then the $\mathcal{E}$-coordinates are the same as ________________________.



(a) Simple monoclinic

(b) Body-centered cubic

(c) Face-centered orthorhombic

FIGURE 3 Examples of unit cells.

The coordinates of atoms within the crystal are given relative to the basis for the lattice. For instance,

$$\begin{bmatrix}1/2\\1/2\\1\end{bmatrix}$$

when representing with respect to the basis $\{u, v, w\}$ refers to

Do you realize what we've just done?

- This might be the most important thing ever discovered about linear algebra.
- Ok, get ready. Here we go:
- Let's take a vector space V and find a basis for V.
- Example: Maybe we have P_2 with basis $\{1, t, t^2\}$
- Now **every** vector $\mathbf{v}$ in V has ______________________.
- So we can _________________ $\mathbf{v}$ with ______________________.

Theorem:
The function:

is a ______________________, that is ___________ and ___________.

This means that every vector space (or subspace) can be thought of as "the same" as $\mathbb{R}^n$ for some n.

Definition: A ____________ transformation

$$T : V \to W$$

that is ____________ and ____________ is called an __________________.

- Spaces that are __________________ can basically be thought of as being "the same"
- We actually use this as motivation for defining **the dimension of a vector space V is n if V is isomorphic to $\mathbb{R}^n$.**

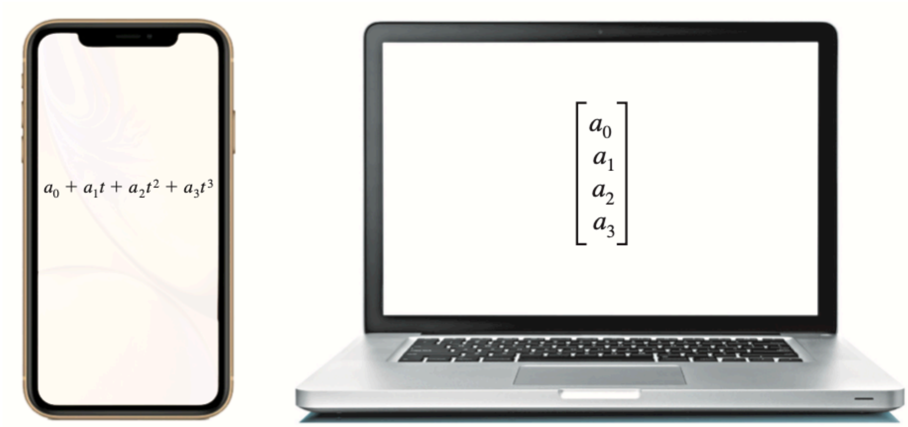


FIGURE 6 The space $\mathbb{P}_3$ is isomorphic to $\mathbb{R}^4$.

Example:

Using the basis $\mathcal{B} = \{1, t, t^2\}$ use coordinates to see if the vectors

$$1 + 2t^2, 4 + t + 5t^2, 3 + 2t$$

are linearly independent.

EXAMPLE 7 Let

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix},$$

and $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$. Then $\mathcal{B}$ is a basis for $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Determine if $\mathbf{x}$ is in H, and if it is, find the coordinate vector of $\mathbf{x}$ relative to $\mathcal{B}$.

SOLUTION If $\mathbf{x}$ is in H, then the following vector equation is consistent:

$$c_1 \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$$

The scalars c_1 and c_2, if they exist, are the $\mathcal{B}$-coordinates of $\mathbf{x}$. Using row operations, we obtain

$$\begin{bmatrix} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $c_1 = 2$, $c_2 = 3$, and $[\,\mathbf{x}\,]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. The coordinate system on H determined by $\mathcal{B}$ is shown in Figure 7. ■

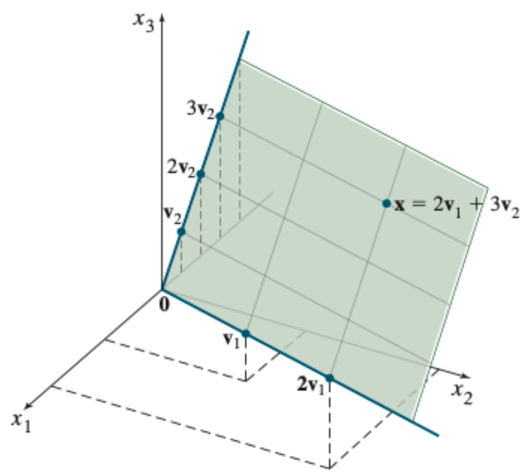


FIGURE 7 A coordinate system on a plane H in $\mathbb{R}^3$.

28 Getting Practice with Coordinates - Transformations

Here are some big questions that we'll focus on today:

- Suppose we have some mysterious vector space V. Suppose that we have two different bases B and C. How do we know that they have the same number of elements? Why can't it happen for example that B has 4 elements and C has only 3.
- Suppose V is 10 dimensional and H is a subspace of V. What are the possible dimensions of H?
- Suppose we have an abstract vector space, like P_3 and we have a linear transformation like

  $$T : P_3 \to P_3, \quad T(p(t)) = p'(t)$$

  Since we can turn polynomials into vectors using coordinates, can we turn transformations into matrices?
- What is the "rank" of a matrix? New definition!

29 The Rank Nullity Theorem

Last time we saw that if we __________________________ for an n-dimensional vector space V then we can describe a linear transformation $T : V \to V$ in terms of coordinates, as a transformation $\mathbb{R}^n \to \mathbb{R}^n$.

The same thing works with transformations:

$$T : V \to W.$$

The process is always:

- Fix a basis for V, $\{b_1, \ldots, b_n\}$
- Fix a basis for W, $\{v_1, \ldots, v_m\}$
- Write out the coordinates of $T(b_i)$ in terms of the basis of W and those will become the standard matrix for T.

Example:

Last time we considered the linear transformation defined by $T : P_3 \to P_3$ given by $T(p(t)) = p'(t)$. The matrix for this transformation was

What is the **range** of T?

What is the **kernel** of T?

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Connection:

Calculate A^2:

$$A^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} =$$

This transformation should correspond to ____________________________.

What is the **range** of T?

What is the **kernel** of T?

Consider the linear transformation $T : P_2 \to M_{2\times2}$ given by

$$T(a + bt + ct^2) = \begin{bmatrix} a - c & b - c \\ 0 & c + b - 2a \end{bmatrix}$$

1. Choose bases for the domain and codomain.
2. Write down the standard matrix for T.
3. Find a basis for the Kernel and Range of T.
4. Find the dimension of the Kernel and Range of T.
5. Is it easier / harder to think in terms of Nul and Col?

Consider the following transformation defined by the matrix A:

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix} \sim B = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

What is the dimension of the Column space of A? Why? In general, what is the dimension of the Null Space?

What is the dimension of the Null space of A? Why? In general, what is the dimension of the Null Space?

Rank Nullity Theorem

Suppose that A is an $m \times n$ matrix. Then

Definition: We define the **rank of** A to be ______________________.

We define the **nullity of** A to be ______________________.

More generally if $T : V \to W$ is any linear transformation then

Let's get some practice:

EXAMPLE 5 Find the nullity and rank of

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

Example: If a 7×9 matrix has nullity 2, what is the rank of A?

Could a 6×9 matrix have nullity 2?

EXAMPLE 8 A scientist has found two solutions to a homogeneous system of 40 equations in 42 variables. The two solutions are not multiples, and all other solutions can be constructed by adding together appropriate multiples of these two solutions. Can the scientist be *certain* that an associated nonhomogeneous system (with the same coefficients) has a solution?

THEOREM

The Invertible Matrix Theorem (continued)

Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

m. The columns of A form a basis of $\mathbb{R}^n$.

n. Col $A = \mathbb{R}^n$

o. rank $A = n$

p. nullity $A = 0$

q. Nul $A = \{\mathbf{0}\}$

33. If a 4×7 matrix A has rank 4, find nullity A, rank A, and rank A^T.

34. If a 6×3 matrix A has rank 3, find nullity A, rank A, and rank A^T.

35. Suppose a 5×9 matrix A has four pivot columns. Is Col $A = \mathbb{R}^5$? Is Nul $A = \mathbb{R}^4$? Explain your answers.

36. Suppose a 5×6 matrix A has four pivot columns. What is nullity A? Is Col $A = \mathbb{R}^4$? Why or why not?

37. If the nullity of a 5×6 matrix A is 4, what are the dimensions of the column and row spaces of A?

38. If the nullity of a 7×6 matrix A is 5, what are the dimensions of the column and row spaces of A?

39. If A is a 7×5 matrix, what is the largest possible rank of A? If A is a 5×7 matrix, what is the largest possible rank of A? Explain your answers.

30 Practice with Rank Nullity and Change of Basis

Last time we saw that if A is an $m \times n$ matrix then

$$\text{rank}(A) + \text{nullity}(A) = n$$

Where

- rank(A) is the dimension of the column space of A
- nullity(A) is the dimension of the null space of A.

Let's get some practice with this:

EXAMPLE 5 Find the nullity and rank of

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

Hint: You are given that this matrix has 2 pivots.

Example: If a 7×9 matrix has nullity 2 what is the rank of A?

Example: Could a 6×9 matrix have nullity 2?

More generally, if $T : V \to W$ is a linear transformation then

$$\dim \ker T + \dim \text{range} T = \dim V.$$

Example: Find these three numbers in the above equation for the transformation given by

$$T : M_{3\times 2} \to P_7$$

$$T\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = (a-b)t^3 + ct^2 + dt + e$$

Hint: Can you write down the general matrix in the kernel of T? What do a, b, c, d, e, f have to be?

Example: A student is studying a linear transformation:

$$T : M_{4\times 5} \to P_8$$

For each of the following, what could you say about the numbers $\dim \ker T$ and $\dim \text{range} T$?

1. T is onto

2. You find two inputs $\mathbf{u}, \mathbf{v}$ in $\ker T$ that are not multiples of each other.

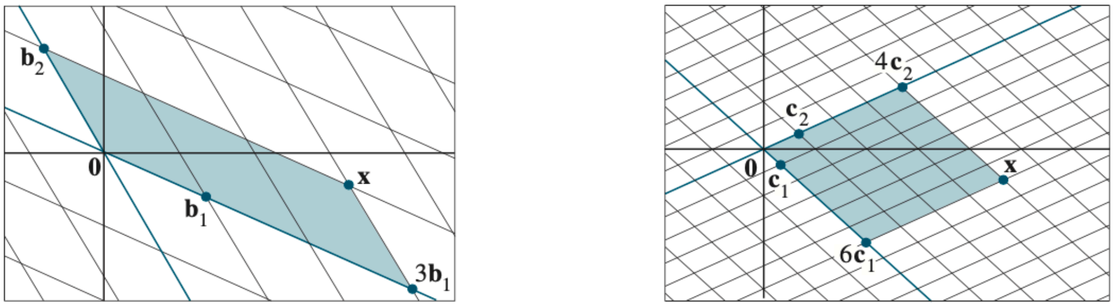
3. The range of T doesn't include any polynomials of degrees 6 or higher.

Change of Basis:

To visualize the problem, consider the two coordinate systems in Figure 1. In Figure 1(a), $\mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2$, while in Figure 1(b), the same $\mathbf{x}$ is shown as $\mathbf{x} = 6\mathbf{c}_1 + 4\mathbf{c}_2$. That is,

$$[\,\mathbf{x}\,]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \text{and} \quad [\,\mathbf{x}\,]_{\mathcal{C}} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

Our problem is to find the connection between the two coordinate vectors. Example 1 shows how to do this, provided we know how $\mathbf{b}_1$ and $\mathbf{b}_2$ are formed from $\mathbf{c}_1$ and $\mathbf{c}_2$.



(a) (b)

FIGURE 1 Two coordinate systems for the same vector space.

EXAMPLE 1 Consider two bases $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ for a vector space V, such that

$$\mathbf{b}_1 = 4\mathbf{c}_1 + \mathbf{c}_2 \quad \text{and} \quad \mathbf{b}_2 = -6\mathbf{c}_1 + \mathbf{c}_2 \tag{1}$$

Suppose

$$\mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2 \tag{2}$$

That is, suppose $[\,\mathbf{x}\,]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Find $[\,\mathbf{x}\,]_{\mathcal{C}}$.

THEOREM 15 Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ be bases of a vector space V. Then there is a unique $n \times n$ matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ such that

$$[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\mathbf{x}]_{\mathcal{B}} \tag{4}$$

The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are the $\mathcal{C}$-coordinate vectors of the vectors in the basis $\mathcal{B}$. That is,

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \cdots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix} \tag{5}$$

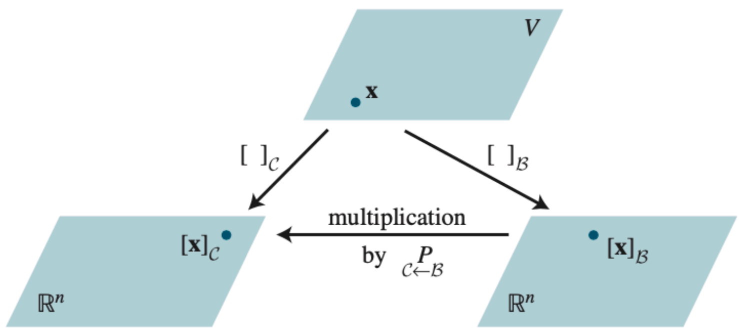


FIGURE 2 Two coordinate systems for V.

Cool Properties:

1. The relationship between $P_{\mathcal{C}\leftarrow\mathcal{B}}$ and $P_{\mathcal{B}\leftarrow\mathcal{C}}$ is

2. If you have three different bases, say $\mathcal{B}, \mathcal{C}, \mathcal{D}$ then

3. In $\mathbb{R}^n$ if $\mathcal{E}$ is the standard basis, then

$$P_{\mathcal{E}\leftarrow\mathcal{B}} = [\mathbf{b_1}, \mathbf{b_2}, \dots, \mathbf{b_n}]$$

By using these ingredients, you can turn most of these problems into a combination of “write down a matrix”, “multiply some stuff”, “take an inverse”.

31 Quiz Day + Catchup with the Change of Basis Stuff

32 Conceptually Thinking about Change of Basis + Dot Product

Last time we saw that if:

- We have two different ________________________
- And we want to convert from ____.
- Then there is a ________________________ that will do the job.
- This is called the ____
- And if we multiply a vector $[x]_B$ by this matrix, we will get $[x]_C$.

In practice, how do we do this?

In practice, we should probably program a machine to do this task. Here's an algorithm that will work:

If you want to find $P_{\mathcal{C}\leftarrow\mathcal{B}}$ then, if you write the big augmented matrix

$$[\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_n} \mid \mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_n}]$$

and row reduce this to RREF, then the matrix you get on the RHS will be $P_{\mathcal{C}\leftarrow\mathcal{B}}$.

This works just ____________. Later we will see **why**.

However, as mathematicians we want to understand more of the "why", so let's try to turn this hard problem:

find $P_{\mathcal{C}\leftarrow\mathcal{B}}$

into an easier problem:

do some (combination) of easier things,

Easiest Version of Change of Basis Matrix

If ever we want to convert from $\mathcal{B}$ to the standard basis $\mathcal{E}$ then our task is very easy:

For example: Find $P_{\mathcal{E}\leftarrow\mathcal{B}}$ where $\mathcal{B}$ is the basis $\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}\}$ of $\mathbb{R}^2$. Check that your matrix will convert the vector $[x]_B = \begin{bmatrix}1\\1\end{bmatrix}$ into its rightful coordinates in the standard basis.

Connecting these Matrices

If we have a matrix that will convert from B coordinates to C coordinates, what do you think the matrix would be that converts from C coordinates to B coordinates?

[Medium/Easy] Combining

If you have three different bases, say $\mathcal{B}, \mathcal{C}, \mathcal{D}$ then

Tada! The big conclusion

By using these ingredients, you can turn most of these problems into a combination of "write down a matrix", "multiply some stuff", "take an inverse".

Let's try it out, and find $P_{\mathcal{C}\leftarrow\mathcal{B}}$ where

$$\mathbf{b}_1 = \begin{bmatrix} 7 \\ 5 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -3 \\ -1 \end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

Last time we worked this out "the hand way" and saw that we got $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}$

We only have 9 days of class left, after this one. What I'd like to do are:

1. Discuss some applications of linear algebra - focusing on "eigenvectors" (this can help us estimate long term behavior of complex systems.
2. Your project will talk about "least squares" regressions which help you approximate solutions to inconsistent systems.

Your project will use something called the "dot product".

If $\mathbf{u}, \mathbf{v}$ are two vectors in $\mathbb{R}^n$ then their dot product, is a scalar, defined in the following way:

If $\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ then

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Example: Find the dot product of $\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$.

Example: Find the dot product of $\mathbf{e}_1 \cdot \mathbf{e}_3$ in $\mathbb{R}^4$.

THEOREM 1

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$, and let c be a scalar. Then

a. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

b. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

c. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$

d. $\mathbf{u} \cdot \mathbf{u} \geq 0$, and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$

Why would anyone want to compute dot products?

- I'm ____________ you ____________ ____.
- Dot products help us calculate ____________ via the formula below:

- Dot products help us calculate ____________ via the formula below:

- This means, for instance, that two vectors are ____________ if and only if

Dot Products to the Rescue

Suppose you are working with a bunch of vectors in $\mathbb{R}^{47}$ and you know that your vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are all **perpendicular** to each other. Suppose you have a mystery vector $\mathbf{w}$ that you know is in $\text{Span}(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3})$.

Reality Check: If we wanted to write $\mathbf{w}$ as a linear combination of the $\mathbf{v}$'s, we could just:
set up a system with ____________ equations and ____________ variables. This would be fine for a computer to do. But we'd have to do a LOT of row operations. Now since you have all these vectors, you can very **quickly** take the dot products of them with each other. Say you get this data:

$$\mathbf{v_1} \cdot \mathbf{v_1} = 1 \qquad \mathbf{w} \cdot \mathbf{v_1} = 3$$

$$\mathbf{v_2} \cdot \mathbf{v_2} = 2 \qquad \mathbf{w} \cdot \mathbf{v_2} = -7$$

$$\mathbf{v_3} \cdot \mathbf{v_3} = 9 \qquad \mathbf{w} \cdot \mathbf{v_3} = 5$$

Use this information to write $\mathbf{w}$ as a linear combination of $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$.

33 Eigenvectors

Remember me?

THEOREM 8 **The Invertible Matrix Theorem**

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.
b. A is row equivalent to the $n \times n$ identity matrix.
c. A has n pivot positions.
d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
e. The columns of A form a linearly independent set.
f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
h. The columns of A span $\mathbb{R}^n$.
i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
j. There is an $n \times n$ matrix C such that $CA = I$.
k. There is an $n \times n$ matrix D such that $AD = I$.
l. A^T is an invertible matrix.

THEOREM **The Invertible Matrix Theorem (continued)**

Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

m. The columns of A form a basis of $\mathbb{R}^n$.
n. $\text{Col } A = \mathbb{R}^n$
o. $\text{rank } A = n$
p. $\dim \text{Nul } A = 0$
q. $\text{Nul } A = \{\mathbf{0}\}$

Let's add two more:

Invertible Matrix Theorem (continued)

r.

s.

Warmup:

Let $A = \begin{bmatrix} 6 & -4 \\ -10 & 9 \end{bmatrix}$.

Consider the questions below. Are they all equally difficult? What could we try?

a.) What is A^{10}?

b.) What is $(A^{10})\begin{bmatrix} 13 \\ 0 \end{bmatrix}$?

c.) What is $(A^{10})\begin{bmatrix} 4 \\ 5 \end{bmatrix}$?

Important Definition

If A is an $n \times n$ matrix, then we say that a vector $\mathbf{v}$ is an ____________________________ if

Check that $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is an eigenvector for the matrix A above. What is its **eigenvalue**? What about $\begin{bmatrix} -3 \\ 6 \end{bmatrix}$?

Is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ an eigenvector for A? Why or why not?

OMG APPLICATION: (Eigenvectors are cool)

Notice that $\begin{bmatrix} 13 \\ 0 \end{bmatrix} =$

Can we use this to go back and answer part b above?

Consider the questions below: Eventually we will know how to answer all of these, but for the moment, which of these seems easiest. How would you answer it?

- Given a matrix - how can we find its eigenvectors?
- Can we find its eigenvalues?
- Given a matrix and a vector can we test if the vector is an eigenvector for that matrix?
- Given a matrix and an eigenvalue can we find the eigenvectors for that particular eigenvalue?

Example: Show that 7 is an eigenvalue of the matrix $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ and find the corresponding eigenvectors.

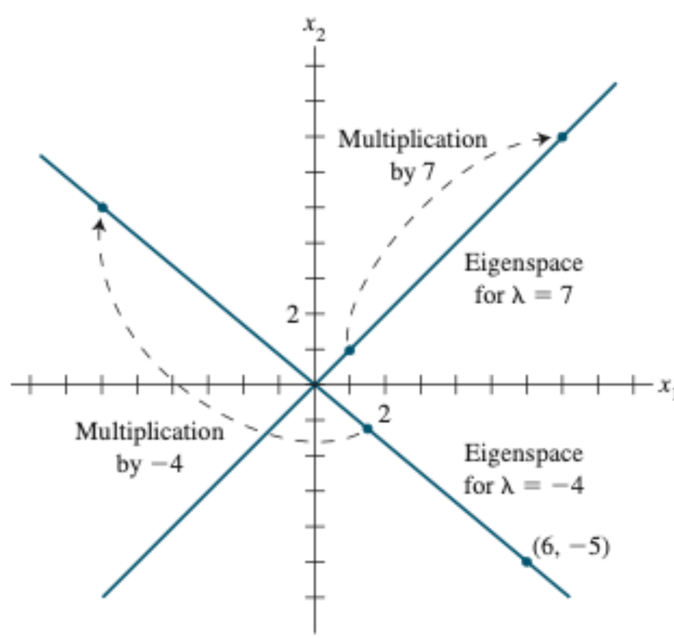


FIGURE 2 Eigenspaces for $\lambda = -4$ and $\lambda = 7$.

Example:

What would it mean if $\lambda = 0$ is an eigenvalue of a matrix A?

Important Definition

If λ is an eigenvalue for a matrix A then

"the ______________ of A ________ corresponding to ________"

is the set of all solutions to the equation:

Equivalently, the eigenspace is the set of all solutions to

Note: The ______________ is always in an eigenspace, but is NOT an eigenvector.
All ______________ vectors in an eigenspace are eigenvectors.

Reality Check

In the previous example, how many eigenvectors did the matrix have?

What was the dimension of the space of eigenvectors with $\lambda = 7$?

What was the dimension of the space of eigenvectors with $\lambda = -4$?

It turns out that for this matrix A, a vector in $\mathbb{R}^2$ is [often / always / rarely / never] an eigenvector.

The eigenvectors are those very special vectors that are not rotated by A, only ________.

Remember, all the other vectors in the plane are being rotated and scaled, in a ________ way.

Example: Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. An eigenvalue for A is 2. Find a basis for the corresponding eigenspace.

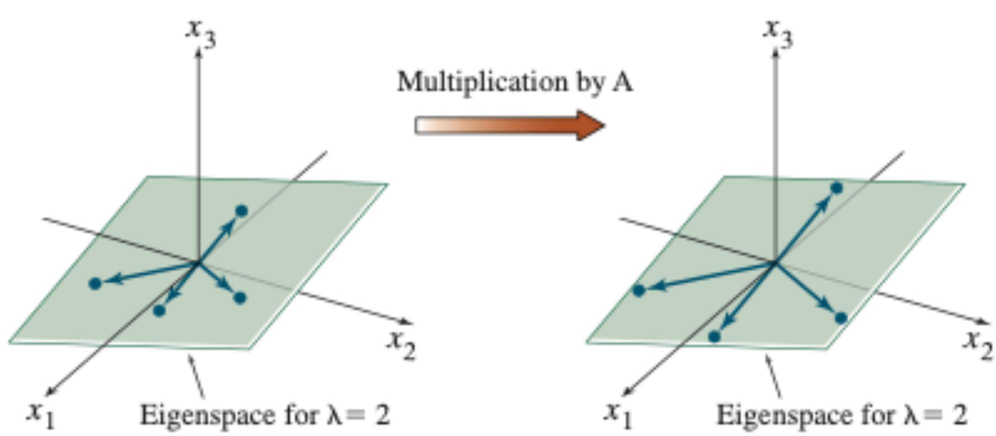


FIGURE 3 A acts as a dilation on the eigenspace.

Three Important Facts

Theorem 1: If λ is an eigenvalue of an $n \times n$ matrix A then

the eigenspace of A corresponding to λ is a ____________ of ____________.

Theorem 2: The eigenvalues of a triangular matrix are ________________________

Theorem 3: If $\mathbf{v_1}, \ldots, \mathbf{v_r}$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A

then the set $\{\mathbf{v_1}, \ldots, \mathbf{v_r}\}$ is ________________________

34 More on Eigenvectors

Let's warm up with some Eigen-trivia

If not stated, A is an $n \times n$ matrix.

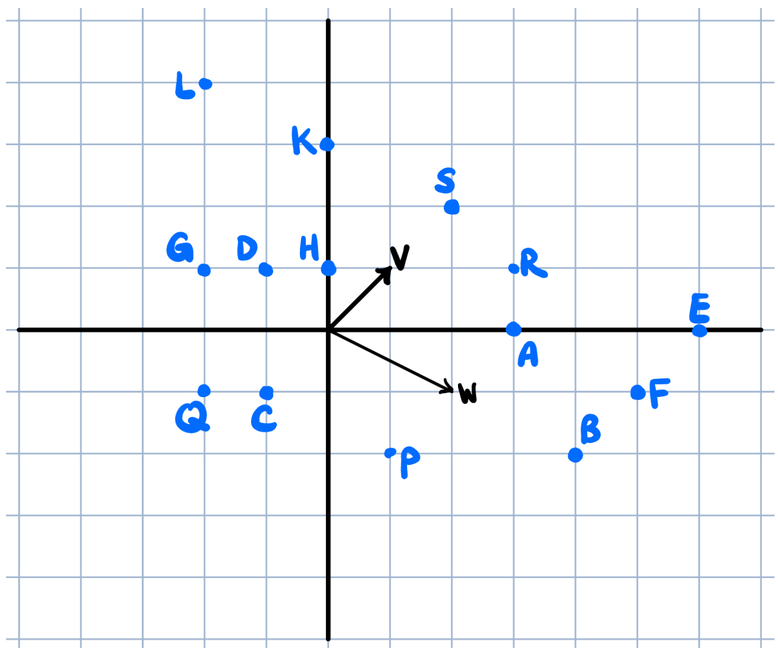
1. The vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is an eigenvector of which of the following matrices (select all that apply)

   (a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

   (b) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

   (c) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

   (d) $\begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix}$



If $\mathbf{v}$ is an eigenvector of A with $\lambda = 2$ and $\mathbf{w}$ is an eigenvector of A with $\lambda = -1$ then

2. which of the pictured vectors is $A\mathbf{v}$?
3. which of the pictured vectors is $A\mathbf{w}$?
4. which of the pictured vectors is $A(\mathbf{v} + \mathbf{w})$?
5. Is $\mathbf{v} + \mathbf{w}$ an eigenvector of A?

   (a) Yes

   (b) No

   (c) We cannot tell

6. If $\mathbf{v}$ is a vector in $\mathbb{R}^n$ such that $A\mathbf{v} = 3\mathbf{v}$ then

   (a) $\mathbf{v}$ is an eigenvector of A with eigenvalue 3.

   (b) $\mathbf{v}$ is an eigenvector of A but we can't tell the eigenvalue.

   (c) $\mathbf{v}$ is NOT an eigenvector of A

   (d) We cannot tell if $\mathbf{v}$ is an eigenvector of A from the given information.

7. If A has fewer than n pivots then

   (a) $\lambda = 0$ could be an eigenvalue

   (b) $\lambda = 0$ must be an eigenvalue

8. Let $A = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix}$

   (a) 1 must be an eigenvalue of A

   (b) 0 must be an eigenvalue of A

   (c) 3 must be an eigenvalue of A

   (d) $1, 3$, and 0 must all be eigenvalues of A.

9. If a matrix A has RREF $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ then

   (a) 1 must be an eigenvalue of A

   (b) 0 must be an eigenvalue of A

   (c) Both 1 and 0 must be eigenvalues of A

   (d) Neither 0 nor 1 must be eigenvalues of A.

10. If $\text{Nul}(A - 7I)$ is a 3-dimensional subspace of $\mathbb{R}^n$ then

   (a) A has a 3 dimensional eigenspace corresponding to $\lambda = 7$

   (b) $A - 7I$ has a 3 dimensional eigenspace corresponding to $\lambda = 7$

   (c) $Nul(A - 7I)$ has a 3 dimensional eigenspace corresponding to $\lambda = 7$

11. If $\mathbf{v}$ is an eigenvector of A with $\lambda = 2$, then what is $A^4(3\mathbf{v})$?

   (a) $16\mathbf{v}$

   (b) $48\mathbf{v}$

   (c) $1296\mathbf{v}$

   (d) $81\mathbf{v}$

12. If $(A-2I)$ is row equivalent to $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

 then which of the following apply (select all that apply)

 (a) A has eigenvalues $1, 3, 0$ and the eigenspace corresponding to $\lambda = 0$ is three dimensional. The others are 1 dimensional.

 (b) A has a 3-dimensional eigenspace corresponding to $\lambda = 2$.

 (c) A is not invertible.

13. If $\mathbf{v}$ is an eigenvector of A with $\lambda = 2$, then

 (a) $\mathbf{v}$ is a always an eigenvector for A^2 with eigenvalue 2

 (b) $\mathbf{v}$ is a always an eigenvector for A^2 with eigenvalue 4

 (c) $\mathbf{v}$ is a sometimes an eigenvector for A^2 with eigenvalue 2

 (d) $\mathbf{v}$ is a sometimes an eigenvector for A^2 with eigenvalue 4

 (e) $\mathbf{v}$ is a never an eigenvector for A^2.

14. If $\mathbf{u}, \mathbf{v}$ is an independent set of two eigenvectors with the **same** eigenvalue λ then

 (a) $\mathbf{u} + \mathbf{v}$ is an eigenvector with eigenvalue λ

 (b) $\mathbf{u} + \mathbf{v}$ is an eigenvector with eigenvalue 2λ

 (c) $\mathbf{u} + \mathbf{v}$ is an eigenvector with some different eigenvalue

 (d) $\mathbf{u} + \mathbf{v}$ is NOT an eigenvector.

 (e) There is not enough information to tell if $\mathbf{u} + \mathbf{v}$ is an eigenvector.

15. If $\mathbf{u}, \mathbf{v}$ are two eigenvectors with the **same** eigenvalue λ then

 (a) $\mathbf{u} + \mathbf{v}$ is an eigenvector with eigenvalue λ

 (b) $\mathbf{u} + \mathbf{v}$ is an eigenvector with eigenvalue 2λ

 (c) $\mathbf{u} + \mathbf{v}$ is an eigenvector with some different eigenvalue

 (d) $\mathbf{u} + \mathbf{v}$ is NOT an eigenvector.

 (e) There is not enough information to tell if $\mathbf{u} + \mathbf{v}$ is an eigenvector.

16. Let $A = \begin{bmatrix} 2 & 1 & -4 \\ 0 & 3 & -4 \\ 0 & 3 & -10 \end{bmatrix}$. You are given that $\lambda = 2$ is an eigenvalue for A. Find a basis for the eigenspace.

17. A has another eigenvalue. How could we find it?

35 Zoom Class From Home (COVID)

36 Some Loose Ends about Eigenvectors

Today our main goals are to:

1. Learn a way to calculate the eigenvalues of an matrix, by using __________________________.
2. Practice some conceptual problems abut eigenvalues for the **quiz on Friday**.
3. Learn some additional theoretical properties about eigenvectors.

Warmup

Let's fill in the logic below.

λ is an eigenvalue for a matrix A $\iff$ the equation ____________________ has a _____________ solution.

$\iff$ the equation ____________________ $= \mathbf{0}$ has a _____________ solution.

$\iff$ the matrix ____________________ has a at least one ____________________.

$\iff$ the matrix _____________ is not __________________________

$\iff$ __________________________.

Way to Find Eigenvalues of Any Matrix:

- The equation you get by setting the determinant equal to zero is called the __________________________.
- the **degree** n **polynomial** $\det(A - \lambda I)$ is called the __________________________.

Remember: The zeros of the characteristic polynomial are the eigenvalues.

Example: Find the eigenvalues of the matrix $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$

Question: If we wanted to find the eigenspaces associated to each eigenvalue what would we do?

Answer:

Example: Find the characteristic polynomial of $A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & 8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

What are the dimensions of the eigenspaces?

Pro tip: The dimension of the eigenspace corresponding to λ is equal to the number of ____________________________ in the matrix $A - \lambda I$.

In this problem, **algebraic multiplicity** of the eigenvalues were:

The algebraic multiplicity of the eigenvalue a is the ______________ of the factor of $(\lambda - a)$ in the characteristic polynomial.

Cool Theorem:

Let A be a matrix and let λ be an eigenvalue of A with multiplicity e. Let E_λ be the eigenspace of A associated to λ. Then

$$\leq \dim E_\lambda \leq$$

Example:

Suppose we have a matrix A whose characteristic polynomial is

$$\lambda^6 - 4\lambda^5 - 12\lambda^4.$$

What can you say about the eigenvalues of A? What can you say about the dimensions of the eigenspaces?

Example:

Find the characteristic polynomial, the eigenvalues and eigenspaces for the matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Does $\mathbb{R}^2$ have a **basis** consisting of eigenvectors of this matrix?

Where are we going next?

- If we are studying an $n \times n$ matrix A then eigenvectors of A are very nice.
- Say we have a 3×3 matrix A. Can we find a basis for $\mathbb{R}^3$ consisting of eigenvectors of A?
- That would be a great situation, because we could then describe **every** vector in $\mathbb{R}^3$ in terms of this basis, and then since A behaves very "nicely" on this **eigenbasis** we now know a lot more about how A transforms **every** vector.

We won't always be able to do this, but we will often enough that it will be very useful for applications.

Useful Lemma (stated for 3 vectors - but it is true in general)

Let A be an $n \times n$ matrix and let $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ be three ______________ of A that have ______________ eigenvalues.

Then

Note: The main way we will apply this is the more general statement that:

If you take a ______________ for each ____________________ of a matrix separately,

then if you put all these vectors together, the resulting set will be a ___________________________ set.

Example:

Suppose you have a 6×6 matrix A with eigenvalues $\lambda = 3, 5, 7$ you are given the following information:

E_3 is 2 dimensional, E_5 is 3 dimensional, E_7 is 1 dimensional.

Question 1: Could you build a basis for $\mathbb{R}^6$ using only eigenvectors of A? If so, how many from each eigenspace would you need?

Question 2: Suppose you were given two eigenvectors $\mathbf{p}, \mathbf{z}$ with $\lambda = 7$. Explain why you know these vectors must be linearly dependent.

Question 3:

You are given 4 nonzero vectors satisfying the equations below:

$A\mathbf{u} = 3\mathbf{u}$

$A\mathbf{v} = 5\mathbf{v}$

$A\mathbf{w} = 5\mathbf{w}$

$A\mathbf{y} = 7\mathbf{y}$

The set $\{\mathbf{u}, \mathbf{v}\}$ is [sometimes / aways / never] linearly independent

The set $\{\mathbf{u}, \mathbf{w}, \mathbf{y}\}$ is [sometimes / aways / never] linearly independent

The set $\{\mathbf{v}, \mathbf{w}\}$ is [sometimes / aways / never] linearly independent

The set $\{\mathbf{v}, \mathbf{w}, \mathbf{y}\}$ is [sometimes / aways / never] linearly independent

Question 4:

Explain why A must be invertible.

37 Diagonalization

What are the eigenvalues (and eigenvectors) for the following matrix?

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

What about this one:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

What are the nth powers of the matrices above?

Important Facts

If A is an $n \times n$ diagonal matrix then the ______________________ are the eigenvalues of A.

Furthermore, the vectors ____________ are all eigenvectors for A.

Therefore, $\mathbb{R}^n$ has a ____________ consisting of ______________________.

Furthermore, finding powers of diagonal matrices is very ____________.

Facts of life:

It would be great if every matrix were diagonal.

However, this isn't true.

This is a fact of life.

Definiton: Let A and B be $n \times n$ matrices. We say that A is ______________________ if

Note: If A is similar to B then B is similar to A. The main point is that we have a ____________ with an invertible matrix.

What the heck - why would we care?

Example:

Suppose that A is similar to B. This means that

$$A = PAP^{-1}$$

What is A^3?

What is A^n?

Powers of similar matrices are similar

Example:

The matrix $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$ is similar to the matrix $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$.

Why do you think I've used the letter D to denote this matrix?

Since A is similar to D this means that *

$$A = PDP^{-1}$$

for some matrix P. Use this information to calculate A^k

* $P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$

Being Similar to a diagonal matrix is pretty useful!

Definition: An $n \times n$ matrix A is called ________________________ if

Here are some sample problems:

- Determine if the matrix $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ is diagonalizable. If so, find the matrices D, P so that

$$A = PDP^{-1}.$$

  Use this to calculate A^k. (This will be much faster, even on a computer to find this decomposition first and then use it to take powers).
- **Real Life Example:** Suppose you have an image file, that is 1000×1000 pixels. Let's say that each pixel takes 1 byte to store, so your image requires 1 million bytes or 1Mb to store.
  You can think of your image as a big 1000×1000 matrix A
  Now what if you did some calculations and noticed that your matrix A could be written:

$$A = P \begin{bmatrix} 87 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 56 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 00.004 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.000000008 \end{bmatrix} P^{-1}$$

  You've got 1000 eigenvalues, but when you analyze them after the first 100, the eigenvalues are all very small, say less than 0.01.
  If you decided to say "Let's just remember the first 100 eigenvalues".
  Then we could store a "compressed" version of our image. We would need to store:
  - The first 100 columns of P (100,000 numbers)
  - The first 100 rows of P^{-1} (100,000 numbers)
  - The 100 eigenvalues (100 numbers)

  This would mean we could compress our image to 200.1kb.

Disclaimer: In practice you would want to do some pre-processing first, and do what is called a "Singular Value Decomposition of your image." This works extremely well.

Quest:

Given a matrix A, determine if A is diagonalizable.

- If so, what are the matrices D, P so that $A = PDP^{-1}$?

Eigenvalues to the rescue

Theorem

If A and B are similar then they have the same ____________________________. As a corollary this means that they have the same ____________________________.

Proof:

Big Theorem: An $n \times n$ matrix A is diagonalizable if and only if A has ______________________________.

In fact if such eigenvectors exist and P is the matrix whose columns are these eigenvectors, then

$$A = PDP^{-1}$$

where D is the diagonal matrix whose entries are the corresponding eigenvalues.

38 We worked more on the diagonalization packet

39 How to Tell if a Matrix is Diagonalizable

Last time we saw an important definition of when a matrix is _________________________.

It means that A is _________________________ to a diagonal matrix.

Warning:

Which of the following are diagonal matrices?

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 5 \\ 0 & 5 & 0 \\ 2 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Big Theorem: An $n \times n$ matrix A is diagonalizable if and only if $\mathbb{R}^n$ has a ____________

consisting of ______________________________________.

In fact if such eigenvectors exist and P is the matrix whose columns are these ___________________, then

$$A = PDP^{-1}$$

where D is the diagonal matrix whose entries are the corresponding ___________________.

Your Field Guide to Diagonalizing a Matrix

To diagonalize a matrix.

1. Find the eigenvalues and the characteristic polynomial. If any of the eigenvalues are not real (meaning they are complex or imaginary) then your matrix is NOT diagonalizable.
2. Even if you have all real eigenvalues, you are still **not sure** whether the matrix is diagonalizable.

   To figure this out you need to find the _________________________.
3. If the sum of the dimensions of your eigenspaces is equal to n, then this means your matrix is diagonalizable and D will consist of the eigenvalues on the diagonal, and P will consist of basis vectors for the corresponding eigenspaces.
4. If the sum of the dimensions of the eigenspaces is less than n, then your matrix is NOT diagonalizable.

Example: You have a 4×4 matrix A and you know the following about it. Use this information to determine if A is diagonalizable, and if so, find D and P.

The characteristic polynomial of A is $(\lambda + 3)^2(\lambda)(\lambda - 5)$.

You know that the eigenspaces have bases:

E_{-3} has basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 4 \\ 0 \end{bmatrix} \right\}$

E_0 has basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

E_5 has basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix} \right\}$

Example: You have a 3×3 matrix A and you know the following about it. Use this information to determine if A is diagonalizable, and if so, find D and P.

The characteristic polynomial of A is $(\lambda^2 + 1)(\lambda - 3)$

Example: You have a 4×4 matrix A and you know the following about it. Use this information to determine if A is diagonalizable, and if so, find D and P.

The characteristic polynomial of A is $(\lambda + 1)^2(\lambda - 3)(\lambda + 5)$

E_{-1} has basis $\{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}\}$

E_3 has basis $\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\}$

E_{-5} has basis $\{\begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}\}$

Example: You have a 4×4 matrix A and you know the following about it. Use this information to determine if A is diagonalizable, and if so, find D.

The characteristic polynomial of A is $(\lambda + 1)(\lambda - 3)(\lambda + 5)(\lambda - 4)$

Important Useful Theorem

Theorem: If A has n distinct real eigenvalues then A is diagonalizable.

Remember this **Theorem** from earlier. The dimension of E_λ will always satisfy

$$1 \le \dim E_\lambda \le \text{the algebraic multiplicity of } \lambda$$

Example:

Suppose you have a matrix A and you know that the characteristic polynomial of A

is $(\lambda - 2)^4(\lambda - 3)(\lambda + 5)$

What can you say about the dimensions of the eigenspaces? What would you need to know to determine if A were diagonalizable?

Example:

Suppose that A is a 3×3 matrix and you know the following about A:

- $A\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = 4\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$
- There are two linearly independent vectors $\mathbf{v}, \mathbf{w}$ that satisfy $A\mathbf{v} = -\mathbf{v}$ and $A\mathbf{w} = -\mathbf{w}$.

Explain why A must be diagonalizable and say as much as you can about D and P.

Determine if the matrix $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ is diagonalizable. If so, determine P and D so that $A = PDP^{-1}$.

We saw before that powers of the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ will have entries given by the **Fibonacci numbers.**

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots,$$

$$f_0 = 0,\ f_1 = 1 \text{ and } f_n = f_{n-1} + f_{n-2}$$

For example:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} f_0 & f_1 \\ f_1 & f_2 \end{bmatrix}, \qquad A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} f_1 & f_2 \\ f_2 & f_3 \end{bmatrix}, \qquad A^3 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} f_2 & f_3 \\ f_3 & f_4 \end{bmatrix}$$

$$A^n = \begin{bmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{bmatrix}$$

So if we can find the nth power of A, we'll know the formula for the nth Fibonacci number.

Last time we found the eigenvalues of A:

$$\lambda_1 = \frac{1+\sqrt{5}}{2}, \quad \lambda_2 = \frac{1-\sqrt{5}}{2}$$

Note:

- $\lambda_1 + \lambda_2 =$

- $\lambda_1 - \lambda_2 =$

- Remember, the characteristic equation was $\lambda^2 - \lambda - 1 = 0$ so both roots satisfy this equation.

In order to minimize our work: Let's find the eigenspace attached to λ in general.

$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\mathbf{x} = \lambda\mathbf{x}$, so we solve $\begin{bmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}\mathbf{x} = \mathbf{0}$. So we need to row reduce:

An eigenvector for λ_1 is and an eigenvector for λ_2 is .

$A = PDP^{-1} =$

Let's calculate $P^{-1} =$

So then remember $A^n = \begin{bmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{bmatrix}$. Let's find the lower left entry of A^n using what we have on the previous page:

$$A^n = \frac{1}{\sqrt{5}} \begin{bmatrix} -\lambda_2 & -\lambda_1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} \begin{bmatrix} 1 & \lambda_1 \\ -1 & \lambda_2 \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \qquad\qquad \begin{bmatrix} 1 & \lambda_1 \\ -1 & \lambda_2 \end{bmatrix}$$

Very often we will **iterate** a process and see how the process **evolves**. For instance, we might look at something like:

$\mathbf{x}_0$

$\mathbf{x}_1 = A\mathbf{x}_0$

$\mathbf{x}_2 = A\mathbf{x}_1$

$\mathbf{x}_n = A\mathbf{x}_{n-1}$

Here are some interesting questions:

Question 1: If we want to know what $\mathbf{x}_n$ is, how much does it matter what the original input $\mathbf{x}_0$ is?

Question 2: Can we say something regardless of the input?

An Application: At a certain school the number of Math, Chemistry and Computer Science majors will fluctuate every year in the following way. Suppose that in a given year

- 20% of the chemistry majors will become math majors, and 10% will become computer science majors. 70% will remain chemistry majors.
- 15% of the math majors will become computer science majors, and 10% will become chemistry majors. 75% will remain math majors.
- 40% of the computer science majors will become math majors, and 5% will become chemistry majors. 55% will remain computer science majors.

If $\mathbf{x}_n = \begin{bmatrix} Ch \\ Ma \\ CS \end{bmatrix}$ represents the numbers of Chemistry, Math and CS majors, then as we go from one year to the next, we have the following evolution:

$$\mathbf{x}_n = \begin{bmatrix} .70 & .05 & .10 \\ .10 & .55 & .15 \\ .20 & .40 & .75 \end{bmatrix} \cdot \mathbf{x}_{n-1}$$

Let's use eigenvalues to figure out what's going on.

EXAMPLE 1 Denote the owl and wood rat populations at time k by $\mathbf{x}_k = \begin{bmatrix} O_k \\ R_k \end{bmatrix}$, where k is the time in months, O_k is the number of owls in the region studied, and R_k is the number of rats (measured in thousands). Suppose

$$\begin{aligned} O_{k+1} &= (.5)O_k + (.4)R_k \\ R_{k+1} &= -p \cdot O_k + (1.1)R_k \end{aligned} \qquad (3)$$

where p is a positive parameter to be specified. The $(.5)O_k$ in the first equation says that with no wood rats for food, only half of the owls will survive each month, while the $(1.1)R_k$ in the second equation says that with no owls as predators, the rat population will grow by 10% per month. If rats are plentiful, the $(.4)R_k$ will tend to make the owl population rise, while the negative term $-p \cdot O_k$ measures the deaths of rats due to predation by owls. (In fact, $1000p$ is the average number of rats eaten by one owl in one month.) Determine the evolution of this system when the predation parameter p is .104.

SOLUTION When $p = .104$, the eigenvalues of the coefficient matrix $A = \begin{bmatrix} .5 & .4 \\ -p & 1.1 \end{bmatrix}$ for the equations in (3) turn out to be $\lambda_1 = 1.02$ and $\lambda_2 = .58$.

Final Exam Study Guide: Last Updated Tuesday November 29th

The final exam will be cumulative, but there will be an emphasis on the material since our midterm. (This makes a lot of sense because everything naturally builds in the course.)

The best way to review is to go back and review old quizzes, worksheets, the midterm. Pay special attention to definitions - the first question on the exam will ask you to give definitions.

On the next page are some problems concerning the final topics of our class (eigenvectors and diagonalization.) These problems are good examples of questions I might ask on the final exam.

Here is a full summary of the topics you should expect to show up on the exam.

Chapter 1

What is an augmented matrix? How can you tell if a system is consistent? If a system has a solution, how do you tell if it is unique? What are pivots? Free variables? What is echelon form? What are linear combinations of vectors? How does the equation $Ax = b$ work? How do you tell if a set of vectors spans $\mathbb{R}^m$? How would you tell if the vectors were linearly independent? What is a linear transformation like "reflect across the y-axis"? How do you write down the matrix for a linear transformation? What does it mean for a matrix transformation to be 1-1? Onto?

Chapter 2

How do you multiply matrices? How do you find the inverse of a 2×2 matrix? What's the algorithm to find the inverse of an $n \times n$ matrix? What are the many ways we can check if an $n \times n$ matrix is invertible? If you know that A is invertible and $Ax = By$ can you solve for x? (Yes, multiply both sides by A^{-1} on the left.)

Chapter 3

Be able to calculate the determinant of a 2×2 matrix and a 3×3 matrix by expanding by rows or columns. Be able to find the determinant of a triangular matrix. Know that the absolute value of the determinant represents the factor by which volume is being stretched when we apply the transformation given by A. Know that the determinant of A^{-1} is $1/\det(A)$ and that $\det(AB) = \det(A)\det(B)$. Know how row operations affect the determinant and be able to "back-solve" to get a determinant if you know the echelon form. Know what the determinant says about a matrix being invertible.

Chapter 4 Be familiar with our main examples of vector spaces: $\mathbb{R}^n, P_n, M_{m\times n}$, and the set of all "functions" like $\sin x$ and e^x. Given some vectors, what is their span? What are some examples of linear combinations? Can you tell if vectors are linearly independent?

What does it mean to be a subspace? (Know the definition, but also how to check. Remember, the first property to check is to see if the zero vector is in the set.) What is the dimension of a subspace? What is a basis? What is the column space and null space of a matrix (know the definitions) and be able to find their bases.

Know the rank-nullity theorem, both for matrix transformations and for general linear transformations:

$$\dim \ker T + \dim \operatorname{range} T = \dim \text{of the domain of } T.$$

Be able to use this formula to answer questions. What is the rank of a matrix? How can you read it off? Can you write the basis for the kernel of a linear transformation? For the range? If a vector space is 3 dimensional, how many vectors will there be in the space? (infinitely many) in the basis? (3).

Be able to answer questions about coordinates with respect to a basis. I won't ask you to calculate a "change of basis matrix" but I might ask you to use one. For instance I might say something like in the problems in this review packet. Be able to write down coordinates of a vector in terms of its basis and know what that represents.

Chapter 5

Given a matrix A can you find the eigenvalues of A? (You can use the characteristic polynomial for the general matrix, but if the matrix is triangular is there an easier way?) Can you check if a given vector is an eigenvector?

Geometrically, what do eigenvectors represent? Given a matrix and an eigenvalue can you find the eigenspace? Can you find a basis for the eigenspace corresponding to an eigenvalue? What is the characteristic polynomial and how can you read off the algebraic multiplicities of the eigenvalues from it? What is the relationship between the dimension of an eigenspace and the algebraic multiplicity of its corresponding eigenvalue? What does it mean for two matrices to be similar? What does it mean for a matrix to be diagonalizable? How can you check if a matrix is diagonalizable? (E.g. if it has n distinct eigenvalues, then the answer is yes.) If $\mathbb{R}^n$ has a basis consisting of eigenvectors of A, does that mean A is diagonalizable? If you were given those vectors and the eigenvalues, could you write down a D, P so that $A = PDP^{-1}$? Given a diagonalized matrix, can you use this to find high powers of the matrix? Be able to interpret information about eigenvectors given conditions on a matrix A, like on the sample problems below)

Optional Written Homework 8: Extra Practice for Final Exam Study

1. A student is working with a vector space and discovers two different bases $\mathcal{B}, \mathcal{C}$. The student really loves the function $\mathbf{v} = e^x + 7\cos x$ which luckily happens to be in their vector space (what are the odds?!!!) Anyways, the student has written down the following information, but has lost their train of thought. Can you help the student complete the missing information:

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \quad P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

   a) What is the dimension of the student's vector space and why?

   b) What is $[\mathbf{v}]_{\mathcal{C}}$?

   c) If you know that $[\mathbf{w}]_{\mathcal{C}} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ then what would $[\mathbf{w}]_{\mathcal{B}}$ be?

   d) What are the $\mathcal{B}$ coordinates for the function $2e^x + 14\cos x$?

2. If an $n \times n$ matrix has n distinct eigenvalues, then it is automatically diagonalizable. For which of the following matrices does this quickly give a positive answer to the question: "is this matrix diagonalizable"

   (a) $A = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$.

   (b) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

   (c) $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$.

   (d) $A = \begin{bmatrix} 1 & 4 & -4 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

   (e) $A = \begin{bmatrix} 1 & 4 & -4 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

   (f) $A = \begin{bmatrix} 1 & 4 & -4 & 5 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ †

---

† answer: the matrices in a,c,d,f all have n distinct eigenvalues, so these matrices are "easily" seen to be diagonalizable.

3. For each of the matrices below, determine if the matrix can be diagonalized. If so, find the matrices D, P so that $A = PDP^{-1}$. If not, the best way to prove this is to show which eigenspace doesn't have dimensional equal to its eigenvalue's algebraic multiplicity.

   I recommend doing these by hand to get a feel for doing the calculations that you will likely have to do on the exam.

   (a) $A = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$

   (b) $A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$ (Hint: this matrix has eigenvalues $\lambda = 1, 2, 3$. This should quickly tell you that the answer is YES, and should quickly tell you what D is. Now you just need to find P. This will take some elbow grease, but after you find your eigenvectors, I recommend double checking with symbolab to see if $A = PDP^{-1}$ is really true.

   (c) $A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$ (Hint: This has eigenvalues $\lambda = 1, 4$)

   (d) $A = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$ (Hint: This has eigenvalues $\lambda = 1, 5$)

   (e) $A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ (Hint: it should be quick to find the eigenvalues of this triangular matrix)

   (f) $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$ (Hint: it should be quick to find the eigenvalues of this triangular matrix)

4. It is important to be able to interpret some conceptual questions about eigenvectors.

   - For instance, remember that if $\lambda = 0$ is an eigenvalue for a matrix A the this means that the equation $Av = 0v$ (or $Av = 0$ has a nontrivial solution.) This means that the matrix A has free variables, and is thus NOT invertible. This is worth remembering - if $\lambda = 0$ is an eigenvalue of a matrix A then this means A is NOT invertible.
   - Another important fact is that the dimension of an eigenspace is always at least 1, and always at most equal to the algebraic multiplicity of the eigenvalue. So for instance, say we were trying to find the dimension of the eigenspace corresponding to $\lambda = 13$ and we found that the dimension of the eigenspace was 2. This means that the algebraic multiplicity of this eigenvalue must be at least 2. So in the characteristic polynomial, we must see $(\lambda - 13)$ with an exponent of 2 or more.
   - Eigenvectors with different eigenvalues are **always** linearly independent. This is really useful when you are building a big basis of eigenvectors. However, eigenvectors from the same eigenvalue may or may not be linearly independent.
   - To tell if an $n \times n$ matrix is diagonalizable, it is sufficient to calculate the dimensions of each eigenspace and show that they sum to n.

   Let's try some problems!

   (a) Suppose that A is an $n \times n$ matrix with determinant zero. Explain why know that A has at least one eigenvalue.

(b) In this class, eigenvalues **must** be real numbers. Not every matrix has an eigenvalue. For instance, check that the matrix below doesn't have any eigenvalues. Do this by showing that the characteristic equation does NOT have any real roots. (This matrix does have **complex eigenvalues** but in this class we focus only on the real numbers)

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

(c) Suppose that the characteristic polynomial of an $n \times n$ matrix A is

$$(\lambda - 3)^2(\lambda - 2)(\lambda - 7)^4.$$

What is n? What are the eigenvalues of A? what is the algebraic multiplicity of each eigenvalue? (note: in this setting you do NOT have enough information to determine whether or not A is diagonalizable)

(d) Suppose that the characteristic polynomial of an $n \times n$ matrix A is

$$(\lambda - 3)(\lambda^2 - 4)(\lambda - 7).$$

What is n? What are the eigenvalues of A? what is the algebraic multiplicity of each eigenvalue? Explain why you know that A must be diagonalizable. If $A = PDP^{-1}$ with D diagonal, what will the diagonal entries of D be? Is A invertible? Why or why not?

(e) Suppose that the characteristic polynomial of an $n \times n$ matrix A is

$$\lambda^2(\lambda - 1)(\lambda - 7).$$

Suppose you are given the **additional information** that the two vectors $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ are in the null space of A. Explain why this means that A is diagonalizable.

Hint: These vectors are giving you some information about an eigenspace. Which one?

(f) Suppose that A is a 7×7 matrix and you are given the following information:

- $A\mathbf{v} = 1\mathbf{v}$ has a solution.
- $A\mathbf{v} = 3\mathbf{v}$ has a non-trivial solution
- $A\mathbf{v} = 4\mathbf{v}$ has infinitely many solutions
- $A\mathbf{v} = 5\mathbf{v}$ has a solution set that is a 3-dimensional subspace of $\mathbb{R}^7$
- A has 5 pivots.

Explain why A must be diagonalizable, and say what the diagonal entries of D must be.

Hint: This one is tough. Think about how each of these items is (or is not) giving you information about the eigenvalues.

Hint: Rank Nullity will help here. (Yes, really!)

Part II

Homework Assignments

1 Written Homework 1

Due Wednesday September 14th (in class)

These written homework assignments give you a chance to work on larger problems and practice your mathematical writing. You will generally be graded upon your presentation and the quality of your write-up. It is ok if there are small errors, but what is important is that all your work is written neatly and professionally. Here are some general tips:

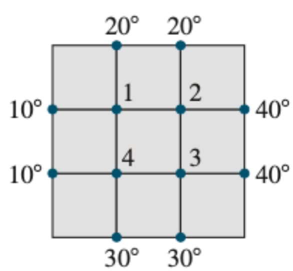
- When writing your solutions, it is very helpful to the reader to start by stating the problem you are trying to solve. This can often be as simple as saying "We are trying to find a solution to ...."
- If there are variables, or letters that you use in your solution, it's important to clearly define them for the reader. When in doubt, consider the word "Let". For instance, you might want to write something like "Let $\vec{v}$ be a solution to the equation $A\vec{v} = \vec{v}$. We are going to investigate ..."
- It is strongly recommended that you do your "scratch work" on a different piece of paper and only include the relevant details in the homework write-up that you submit.
- I recommend you write up your solutions each on a separate page. That way if you need more room you can always insert an additional page.

1. **Application: Heat flow**

   Below is a problem from our textbook about an application involving linear systems of equations.

   An important concern in the study of heat transfer is to determine the steady-state temperature distribution of a thin plate when the temperature around the boundary is known. Assume the plate shown in the figure represents a cross section of a metal beam, with negligible heat flow in the direction perpendicular to the plate. Let $T_1, \ldots, T_4$ denote the temperatures at the four interior nodes of the mesh in the figure. The temperature at a node is approximately equal to the average of the four nearest nodes—to the left, above, to the right, and below.[2] For instance,

   $$T_1 = (10 + 20 + T_2 + T_4)/4, \quad \text{or} \quad 4T_1 - T_2 - T_4 = 30$$



   (a) Write down a system of equations in the variables T_1, T_2, T_3, T_4 that whose solution gives estimates for the temperatures. Please clear denominators so that your equations all have integer coefficients.

   (b) Write down an augmented matrix for your system and calculate its RREF. (You may use a computer to do this, although if you want extra practice you might want to try to do it on your own and check with the computer. Symbolab will do this calculation.)

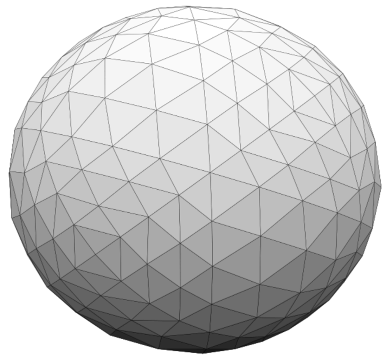
   (c) Using the RREF write down your solution.

   (Remember this assignment is about writing, so make sure your presentation is complete - e.g. if you use a computer, you will want to write something like "I used .. and the result was that this is the .. "

2. **Application to Discrete Math and Geometry**

One beautiful fact about Geometry is as follows: If you take a sphere, and **triangulate it** - that is, you draw a bunch of triangles on it, so that every part of the sphere is in a triangle - then you will come up with some number of faces, edges and vertices, denoted f, e, v respectively. A theorem in topology and geometry says that the following two equations must be true:

$$v - e + f = 2, \quad 3f = 2e$$



(a) Encode these equations in an augmented matrix (and use the alphabetical order e, f, v for your variables. **This order is important - it will affect how your matrix gets written down.**)

(b) By hand, reduce this matrix into Reduced Echelon form and determine where the pivots are.

(c) Suppose you draw a design on a sphere that has 100 faces. How many vertices and edges will you have?

3. **Application to Chemistry**

In Section 1.6 you can read about how linear equations are used to balance chemical equations. Please read this section of the book and use it to balance the equation:

$$B_2S_3 + H_2O \rightarrow H_3BO_3 + H_2S$$

4. **Some mathematical explanations and proofs**

Below are two different problems. The goal is to clearly and carefully provide explanations of these statements.

(a) Suppose that the 3×5 **coefficient** matrix for a system has three pivot columns. Explain why the system HAS to be consistent. Here I will be looking for a solid explanation. You may want to support your answer with a picture of a matrix, or by making explicit reference to specific columns.

(b) A student is solving a system with 3 equations and 9 variables. Explain why IF this system is consistent THEN there must be infinitely many solutions. (Hint: how many pivots can there be?)

2 Written Homework 2

Due Wednesday September 28th (in class)

1. **Getting practice with Pivots, Span, Independence**

   These problems are all good practice for the Quiz we will have on September 23.

   (a) For each of the following situations, write down an example of a 3×3 matrix A and a vector $\vec{b} \in \mathbb{R}^3$ such that the equation $A\vec{x} = \vec{b}$ has

   a. A unique solution

   b. No solutions

   c. A solution space where there is 1 free variable (we think of this space as being **one-dimensional** because it has one degree of freedom.

   d. A solution space where there are 2 free variables (we think of this space as being **two-dimensional** because it has two degrees of freedom.

   e. A solution space where there are 3 free variables (we think of this space as being **three-dimensional** because it has three degrees of freedom.

   (b) For each of the following I want you to write down a set of vectors that has the required property.

   i. A set of 2 vectors in $\mathbb{R}^3$ that are linearly dependent such that all their entries are non-zero.

   ii. A set of 2 vectors in $\mathbb{R}^2$ that span all of $\mathbb{R}^2$ such that the two vectors each have one positive coordinate and one negative coordinate.

   (c) Let A be a 12×6 matrix. You are given the following information about it.

   "There is a vector $\mathbf{b}$ such that the equation $A\mathbf{x} = \mathbf{b}$ has a **unique** solution."

   Now answer the following questions:

   i. The vector $\mathbf{b}$ lives in $\mathbb{R}^?$ (fill in the question mark)

   ii. Are the columns of A linearly independent? Explain your answer.

   iii. Do the columns of A span all of $\mathbb{R}^{12}$? Explain your answer.

2. **Our First Proofs**

These problems are designed to help you improve your proof-writing. Please check out the Videos folder in our google drive for examples of me writing through some similar proofs.

> Will I always give you these proof structures? Not always - but I want to give you these structures at the beginning because this is probably your first time writing this sort of proof. On Midterms and/or the final, you might be asked to prove things and you will not be given the format. You can view these problems as a way to get lots of practice with your proofs!

(a) If $\{\mathbf{v}, \mathbf{w}, \mathbf{u}\}$ are vectors in $\mathbb{R}^4$ and $\mathbf{w} = \mathbf{0}$, then explain why the set $\{\mathbf{v}, \mathbf{w}, \mathbf{u}\}$ is a linearly dependent set. Hint: To show this set is linearly dependent, all you have to do is write down a non-trivial linear combination of the vectors that gives you zero. Can you do it?

> How to write? I recommend writing something like: "I am given that ...... I will show that the set ... is linearly dependent. I will do this by writing down a non-trivial linear combination of these vectors that gives me the zero vector. Here it is: ... This is a non-trivial combination because ....

(b) Suppose that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly independent set of 3 vectors in $\mathbb{R}^n$. (Video will be helpful!)

Prove that the set $\{\mathbf{u} + \mathbf{v}, \mathbf{w} + \mathbf{u}, \mathbf{u} + \mathbf{v} + \mathbf{w}\}$ is a linearly independent set.

(c) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the function defined by

$$T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 2x - 2y \\ 3|y| \end{bmatrix}.$$

Prove that T is NOT a linear transformation by giving explicit data (meaning vectors and/or scalars) for which the properties in the definition are not satisfied.

> How to write: "I will show that the function defined as T blah = blah is NOT a linear transformation. I will do this by showing that it fails property XX of the definition because if I let blah = blah and blah = blah ... then I see that the LHS is .... but the RHS is .... so therefore we do not have property X."

(d) Let $T : \mathbb{R}^p \to \mathbb{R}^q$ be a linear transformation and let $\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ be a linearly dependent set in $\mathbb{R}^p$. Prove that the set $\{T(\mathbf{w_1}), T(\mathbf{w_2}), T(\mathbf{w_3})\}$ is a linearly dependent set of vectors in $\mathbb{R}^q$. (Video will be helpful!)

(e) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Show that if T maps two linearly independent vectors $\mathbf{v}$ and $\mathbf{w}$ onto a linearly dependent set then the equation $T(\mathbf{x}) = \mathbf{0}$ has a nontrivial solution.

> Hint: You know that the set $\{T(\mathbf{v}), T(\mathbf{w})\}$ is linearly dependent. So this means you know that there are weights c_1 and c_2 (not both zero) such that
>
> $$c_1 T(\mathbf{v}) + c_2 T(\mathbf{w}) = \mathbf{0}.$$
>
> Can you use the properties of linearity to find a solution to $T(\mathbf{x}) = \mathbf{0}$.

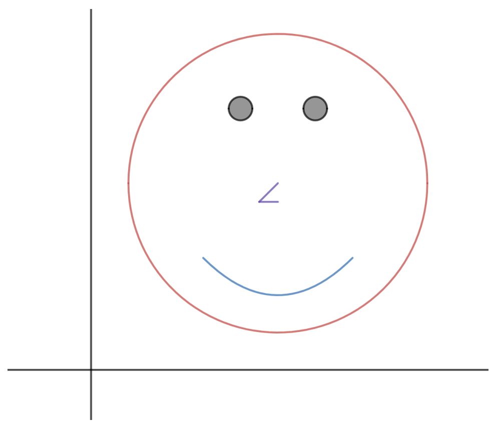
3. **Some Problems about Linear Transformations**

(a) Let A be a 7×10 matrix. Let T be the transformation determined by $T(\vec{x}) = A\vec{x}$. Fill in the blanks and say what the domain and co-domain of T is:

$$T : \mathbb{R}^{blank} \longrightarrow \mathbb{R}^{blank}.$$

(b) We learned that a function $T : \mathbb{R}^n \to \mathbb{R}^m$ is called **a linear transformation** if it satisfies two properties. Write those properties down.

(c) For each of these matrices draw the effect of this matrix on the smiley face. Hint: you can start by seeing where the standard basis vectors $\vec{e_1}$ and $\vec{e_2}$ go.



A.) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

B.) $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

C.) $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

D.) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

E.) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

F.) $\begin{bmatrix} 1/3 & 0 \\ 0 & 2 \end{bmatrix}$

(d) For each of the following descriptions, write down a matrix A so that multiplying by A will have the desired effect.

A.) Rotation by 90 degree counter clockwise

B.) Reflection across the y-axis

C.) Reflection across the x-axis

D.) Reflection across line $y = x$

E.) Reflection across line $y = -x$

F.) First reflect across the line $y = x$ then reflect across the x axis.

G.) First reflect across the x axis and then reflect across the line $y = x$. (you should get a different matrix than in part F before! This shows that the order of composing transformations really matters.)

3 Written Homework 3

Hey everyone, I want to give you ample practice problems for our upcoming midterm. But I don't want you to be swamped with work the same week that everything is due. So here's the plan:

This coming Friday October 7th - there will be NO QUIZ in class. Instead, Part 1 of this HWK will count as your Quiz. Please take care to write up your solutions, neatly and clearly.

Part 1 is Due Friday October 7th in Class

Parts 2 and 3 are Due on Friday October 14th in Class this is the day of the test.

Part 1: These problems are designed to give you practice with doing computations, coming up with examples and with solving equations.

A. **An Application:** At a certain school the number of Math, Chemistry and Computer Science majors will fluctuate every year in the following way. Suppose that in a given year

- 20% of the chemistry majors will become math majors, and 10% will become computer science majors. 70% will remain chemistry majors.
- 15% of the math majors will become computer science majors, and 10% will become chemistry majors. 75% will remain math majors.
- 40% of the computer science majors will become math majors, and 5% will become chemistry majors. 55% will remain computer science majors.

We can encode all of this business into a matrix:

$$\begin{bmatrix} .70 & .05 & .10 \\ .10 & .55 & .15 \\ .20 & .40 & .75 \end{bmatrix} \cdot Current \begin{bmatrix} Ch \\ Cs \\ M \end{bmatrix} = NextYear \begin{bmatrix} Ch \\ Cs \\ M \end{bmatrix}$$

Let's wrap our heads around what's going on: Let's work out the top row of this equation:

.70(# of Chem Majors) + .05(# CS majors) + .10(# Math Majors) = # of Chem Majors Next Year.

- Make sure you understand how this fits together. Do you see how the entries got put into the matrix?
- Now suppose in year 0 there are 40 majors (each) in Chemistry, CS and Math. How many would there be after 1 year. Do this by hand by multiplying your matrix by the vector $\begin{bmatrix} 40 \\ 40 \\ 40 \end{bmatrix}$ - you should get whole numbers for this step.
- Let's do another year. Take your new numbers of majors and multiply by the matrix again. This time you might get decimals. Write down your Year 2 values.
- Now if you wanted to figure out how many majors there will be in 5 years, you could do this:

$$A(A(A(A(A\begin{bmatrix} 40 \\ 40 \\ 40 \end{bmatrix}))))$$

  which is the same as $A^5(\begin{bmatrix} 40 \\ 40 \\ 40 \end{bmatrix})$. This shows the important of multiplying matrices, in particular in taking powers of matrices. Use symbolab (you can type SHIFT caret to do an exponent of a matrix) and see how many of each majors there will be after 5 years.
- Compare this answer with the number of majors after 10 years?
- Symbolab struggles to do A^{20} but what do you think would happen after 20 years? Do you think things would stabilize?
- In fact, if there were an Equilibrium amount of each major, so $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, think about why it would have to satisfy

$$A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

with the extra condition that $x_1 + x_2 + x_3 = 120$ (since we have a total number of 120 students. (In this analogy, I guess no one ever graduates :)) Write down and solve this system of equations. You can use Symbolab to solve the system. Your answer should match (or be very close) to the behavior you got in previous parts.

- You just solved what is called a "dynamical system." You also just found what is called an "eigenvector." We'll learn more about this in 5 weeks.

> Some questions to ponder: What if on the midterm you had a similar setup, but I asked you to write in the entries in the matrix? Do you undersatnd what is going on with the "equilibrium" question? Could you write down the system of 4 equations that you put into Symbolab?

B. You overhear the following conversation between Alice and Bob:

"Hey Bob, you know if I rotate $\mathbb{R}^2$ by and angle α and then rotate it by an angle β, then shouldn't that just be the same as rotating it by a total angle of $\alpha+\beta$."

"Yes - that sounds right to me."

"But you know, that should mean that if I write down the matrix A for "rotate by α" and B for "rotate by β", then if I multiply BA that should be the matrix for the composition of "do A first then B."

"Yes, and you remembered that the "first operation" is always the matrix on the right, since we apply functions inside out, like e.g. $f(g(x))$, the $g(x)$ happens first. But anyway, you were saying?"

"Well you know I multiplied out BA and I was expecting to get

$$\begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

since you know, that's what my teacher said the matrix for "rotate by $\alpha+\beta$" should be. But I got something different. Look at what I got."

"Oh wow, that looks mighty different - but actually you just re-discovered (and gave a proof of!) some trigonometric identities that I remembered learning in school...."

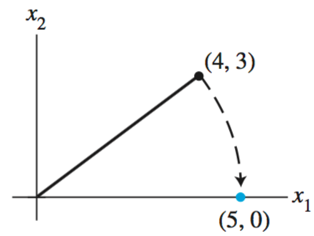
> For this problem do what Alice did and work out the product of matrices as she describes.
> You will discover formulas for $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$ when you identify your matrix with the one above.
> Take care to very carefully label everything in your work.

C. Using the algorithm from Section 2.2, find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Check your answer by multiplying by A and seeing that you get the identity matrix. If you want more practice with this sort of thing - please check out the problems in the textbook. (I've put a pdf of the textbook Ch1 and Ch2 in the google drive for easy access.)

D. You are given a matrix $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ and are told that $a^2 + b^2 = 1$. This matrix is called a Givens Rotation and rotates the plane by some angle (depending on a and b.) For example, in class we've seen the example where $a = 0$ and $b = 1$ come up many times.) In this problem find what a and b need to be so that multiplying by this matrix will send $(4, 3)$ to the point $(5, 0)$. This problem is asking you to solve a matrix equation. (Please try to solve this by thinking about what it is asking and setting up the system yourself. You can probably Google this and find a solution - but will that be helpful? On the test, what if you are given a problem of a similar flavor where you have to struggle and try to figure out what the question is asking? The purpose of homework is to get practice with the struggle and make progress towards a solution.



A Givens rotation in $\mathbb{R}^2$.

E. In this problem you will find lots of examples of matrices with properties. There's no way to do this other than just diving in and trying different numbers until you find something that works. Mathematics is largely about creativity, so don't be afraid to try different things. My recommendation is to start simply (lots of 0s and 1s) to make your life easier before trying bigger numbers.

(a) Find two 2×2 matrices A and B so that $AB = 0$ but $BA \neq 0$. (Here when I say $= 0$, I mean the 2×2 zero matrix).

(b) Find a nonzero 2×2 matrix C such that $C^2 = 0$.

(c) Find two 2×2 matrices D and E so that D and E are both invertible but $E + D$ is NOT invertible.

(d) Find two 2×2 matrices F and G so that F and G are not invertible but $F + G$ IS invertible.

Part 2) These problems are designed to give you some practice with writing explanations and proofs. On the midterm I might ask you to give explanations justifying these sorts of problems.

A. Suppose that A is an $n \times n$ invertible matrix and B is an $n \times n$ matrix (not necessarily invertible). You are analyzing the expression

$$ABA^{-1}$$

for some matrix B and you wonder if it is equal to B. (You think that maybe the A and the A^{-1} cancel, but you know to be suspicious because matrix multiplication is a strange thing, and not commutative.) Show that IF $ABA^{-1} = B$ then this means that $AB = BA$. In other words this will be $ABA^{-1} = B$ only if A and B commute. (Hint: your explanation should be really short. Think about what happens if you "multiply both sides" by something.)

B. In these problems use the invertible matrix theorem to answer these questions. I will be looking for clear and correct explanations. These problems are good examples of the types of questions I might ask on the midterm.

(a) Can a square matrix with two identical columns be invertible? Why or why not?

(b) If A is invertible then the columns of A^{-1} are linearly independent. Why?

(c) If the $n \times n$ matrices E and F have the property that $EF = I_n$ then explain why $EF = FE$. (Use the Invertible matrix theorem)

(d) If A is invertible, is A^2 invertible? If so, what is the inverse?

(e) Explain why the columns of A^2 must be linearly independent if the columns of A are linearly independent.

C. Let A, B be $n \times n$ matrices. Show that if AB is invertible then so is B. Your explanation should say what the inverse of B is.

I'll get you started.
Proof: We know that AB is invertible, so let's call its inverse C. I know then that

$$C(AB) = \underline{\qquad\qquad}.$$

[Now you are actually almost done. From the equation you have above, you should be able to 'see' what the inverse of B is. (Well you might need some help from the trusty invertible matrix theorem.)

Part 3) Some practice with longer proofs. These problems are designed to give you some practice with longer proofs and practice with the 1-1 and onto properties. You might be asked to justify parts of these proofs on the midterm.

0. (Optional, answers in footnote) Which of the following functions are 1-1? Which are onto? Explain your answers. This is to give you practice thinking about 1-1 and onto. Note that **Z** denotes the set of integers. (positive and negative whole numbers).

   (a) The function: $f:\{\text{Students at USD}\}\to\mathbf{Z}$ given by $f(x)=x$'s age in years.

   (b) The function $c:\{\text{Red, Blue, Green, Yellow, Brown}\}\to\{\text{R,B,G,Y}\}$ given by $c(x)=$ the first letter of x.

   (c) The function $C:\{\text{Red, Blue, Green, Yellow, Brown}\}\to\{\text{R,B,G,Y,P}\}$ given by $C(x)=$ the first letter of x.

   (d) The function $g:\mathbb{R}\to\mathbb{R}$ given by $g(x)=3x-2$

   (e) The function $h:\mathbb{R}\to\mathbb{R}$ given by $h(x)=x^3-x$

   (f) The function $r:\{2\times 2\text{ matrices}\}\to\{2\times 2\text{ matrices}\}$ given by $r(A)=A^2$.

   (g) The transformation $T:\mathbb{R}^2\to\mathbb{R}^2$ that is "rotate by 90 degrees clockwise then reflect across the x-axis."

   (h) The linear transformation S whose standard matrix is 5×4 has the maximal number of pivots possible.

1. Suppose that $f:X\to Y$ and $g:Y\to Z$ are 1-1 functions. (Not necessarily linear transformations, but just 1-1 functions.) By watching the video in the google drive, write down a proof that the composition $g\circ f$ is also 1-1. (This is asking you to watch the video and write down the proof and understand what it is saying.)

2. Using your answer from part 1, explain why the following statement MUST be true. I am looking for you to interpret the result from the previous problem and explain why it implies this statement.

   "Suppose, for the moment, that A is an $m\times n$ matrix and B is an $n\times p$ matrix.
   Then if A has n pivots and B has p pivots then the matrix AB has p pivots."

3. It is also true that the composition of onto functions is onto. (This might appear in a future homework assignment). Use this fact to write an analogous statement like we did in the previous problem. Think about what it would say and formulate it carefully with an explanation of why it is true. This is your chance to think critically about what these properties are saying.

Remember - I'm really happy to help. If you can't make my scheduled office hour times, just send me an email and I can set up a "floating office hour" for you to attend.

Answers: a) not 1-1 since some students have the same age, not onto, since no one is 2000 years old e.g. b) Not 1-1 since Blue and Brown are two different inputs that both go to the same output B. It is onto because every think in the codomain is achieved as an output. c) Not 1-1 for the same reason. Not onto since nothing goes to P. d) this function is 1-1 and onto. Every real number can be achieved, (just imagine solving for x), it is 1-1 because there's no way for $3(blah)-2$ to equal $3(foo)-2$ unless $blah=foo$. e) is not 1-1 since e.g. $h(0)=h(1)$. It is onto, by looking at the graph. Every y value occurs. f) This function is not 1-1 since e.g. in part 1 Db on the homework you found a nonzero matrix that squared to give you 0, and of course the zero matrix will also square to give you exactly zero too. For extra credit, work out whether or not this is onto and come present your solution during office hours! g) This function is 1-1 and onto since we can reverse all the steps. This means the function is invertible. Invertible linear transformations are 1-1 and onto. h) Since the matrix will have 4 pivots, there is a pivot in every row, but not every column. This means that the transformation is onto but not 1-1.

4 Written Homework 4

Due: Monday October 25th

Some Problems about Determinants

1. Suppose that a student takes a matrix B and performs the following row operations:
   - Switches rows 1 and 3
   - Replaces row 3 with $R_3 + 2R_1$
   - Replaces row 3 with $\frac{1}{2}R_3$
   - Replaces row 2 with $R_2 - 2R_3$
   - Replaces row 2 with $5R_2$.

   and obtains the matrix
   $$\begin{bmatrix} 1 & 0 & 10 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$
   Calculate the determinant of the original matrix B.

2. Let $B = \begin{bmatrix} 6 & -3 & 2 \\ 0 & 5 & -5 \\ 3 & -7 & 8 \end{bmatrix}$.

   The determinant of this matrix is 45. In this problem you will find the determinant in three different ways, all by hand.

   (a) Expand across row 1. Write your answer very neatly. You might want to just devote a whole page for each part.

   (b) Expand across column 2.

   (c) Use row operations similar to what you did in problem 1. Your first row operation should be to divide row 2 by 5, to help with the calculations.

   Note: You may find this problem challenging - it can be difficult to keep track of everything. A correct solution will have everything clearly laid out so that I can see that you understand how using row operations is related to calculating determinants.

3. One of the most important facts about determinants is that if A and B are $n \times n$ matrices then
   $$\det(AB) = \det(A)\det(B).$$
   Use this fact to find the determinant of ABC is you know the following about A,B,C:
   - All three matrices are 5×5
   - $\det(A) = 10$
   - $BC = 2 \cdot I_5$.

4. It is FALSE that
   $$\det(A + B) = \det(A) + \det(B).$$
   Write down two matrices A and B (of any size you like) that show this second equality is false.

5. Another very important property of determinants is the following:

   An $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$.

   The reason for this is explained in a video I posted in our shared folder. Please watch the video and write down this proof.

6. In this problem I want you to prove: "If A is invertible, then $\det(A^{-1}) = \frac{1}{\det A}$." Do this by filling in the blanks below.

   "Proof: Suppose A is invertible, then this means that the inverse A^{-1} exists and

$$A \cdot A^{-1} = \underline{\qquad\quad}.$$

   Now I will take the determinant of both sides:

   Now using the property that the determinant of a product is equal to the product of the determinants that we saw in Problem _____________ we see that ...

   [complete the rest of the proof. You should end with a solving for $\det(A^{-1})$.]

7. Give a brief explanation for the following statements:

   - If $\det(A^3) = 0$ then A is not invertible. (Your proof should start with "Suppose that $\det(A^3) = 0$" and should end with "Therefore A is not invertible".)
   - Let A and P be $n \times n$ matrices. Show that

$$\det(PAP^{-1}) = \det(A).$$

     (On this problem you may use the result from 6) below).

8. (Extra Credit - To Receive Extra Credit you Must Present your Solution During Offices Hours) If you want to solve this problem, please send your solution to me via email.

   Let R be the triangle in the plane with vertices (x_1, y_1),(x_2, y_2),(x_3, y_3). Show that

$$Area(R) = \frac{1}{2}\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$$

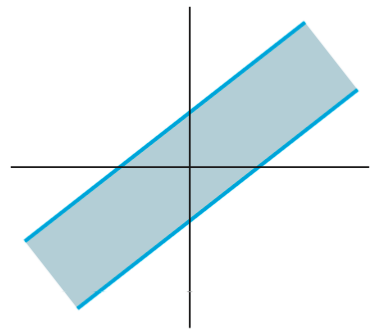
   (Hint: If you translate your triangle so that one vertex is at the origin, can you find the coordinates of the other points? If you do that then can you use determinants to find the area of the parallelogram spanned by those vectors and use that to get the area of the triangle? This is tough but you can do it!)

5 Written Homework 5

Due: Friday October 28th

Is it a subspace?

1. Consider the subset of $\mathbb{R}^2$ shown to the right. Is it a subspace of $\mathbb{R}^2$? Why or why not?



2. Explain why the set

$$H = \{at^3 + 2t^2 + b\ ,\quad a, b \in \mathbb{R}\}$$

is NOT a subspace of P_3.

3. Write a proof that the set of all vectors $\begin{bmatrix} a \\ b \end{bmatrix}$ with $a, b \geq 0$ is NOT a subspace of $\mathbb{R}^2$.

4. Let

$$A = \begin{bmatrix} 3 & 3 & 4 \\ 0 & 1 & 4 \\ -1 & 7 & -2 \end{bmatrix}$$

and let $\vec{b} = \begin{bmatrix} 6 \\ 1 \\ 6 \end{bmatrix}$. Explain why the set of solutions to $A\vec{x} = \vec{b}$ is not a subspace of $\mathbb{R}^3$. (Hint: one property should be very quick to check.)

Problems about Independence

5. Consider the set of vectors in P_3 below:

$$\{1 - t, t^2 - t, t^3 - t^2, 1, t^3\}$$

Determine whether these vectors are linearly dependent. If they are, then find a non-trivial linear combination of them that equals the zero polynomial. Remember, you're all pros at solving systems of equations! You may need to roll up your sleeves and set up some systems of equations and row-reduce or get out symbolab. for this one. Remember that if a polynomial is the zero polynomial then ALL the coefficients must be zero.

6. The next few problems are about the vector space V of all real valued functions $f : \mathbb{R} \to \mathbb{R}$, where addition and scalar multiplication are the usual operations.

   (a) The function $\sin(x + 2)$ is actually in the span of $\sin x$ and $\cos x$. That is there is a way to write:

   $$\sin(x + 2) = c_1 \sin(x) + c_2 \cos x.$$

   Figure out what the coefficients are and check your solution on desmos.com. You might need to look up some old trig formulas.

   (b) Is $\sin(2x)$ in the span of $\sin x$ and $\cos x$? (Answer: no) prove this. Hint: If $\sin(2x) = c_1 \sin x + c_2 \cos x$ really were true, then then this would need to be true for all values of x. Try a few values of x, say $x = 0$ and $x = \pi/2$ and see what you get! You should get some very explicit information about c_1 and c_2. Remember this hint. It might be useful in later homework or quizzes. You are getting practice proving that functions are linearly independent.

(c) Use the method of the previous problem to prove that the function e^x is not in the span of $\{\sin x, \cos x\}$.

(d) Which of the following are in $\text{span}\{\sin^2 x, \cos^2 x\}$?

$$f(x) = 1, \quad g(x) = 3x^2, \quad h(x) = \cos x, \quad p(x) = \cos(2x).$$

For answers where you said yes, write the function as a linear combination. For the answers where you said no, you don't have to prove your answer.

Some quick Constructions

7. Construct a nonzero 3×3 matrix A and a nonzero vector $\vec{b}$ so that b is in Col(A) but $\vec{b}$ is not the same as any one of the columns of A.

8. Construct a nonzero 3×3 matrix A and a nonzero vector $\vec{b}$ so that b is NOT in Col(A). Hint: Can you come up with a very simple example?

9. What can you say about Nul (B) when B is a 5×4 matrix with linearly independent columns. (Hint: how many free variables are there?) **Remember** that Null space is just a synonym for Kernel and Column Space is a synonym for Range in the case of matrix transformations.

Some Extended Calculations

10. Solve the following system of equations: $\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 3 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Is your answer a subspace of $\mathbb{R}^3$? If so, write it as the span of a set of vectors.

11. Consider the two transformations below. Only one is linear. Which one?

$$S : P_3 \to P_3, \qquad S(p(t)) = tp'(t) - p(t)$$

$$F : P_3 \to P_6, \qquad F(p(t)) = (p(t))^2$$

More specifically, I want you to

- For the function that is NOT linear, explain why by giving an explicit violation of the definition. You should write something like: I will show the property $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ is NOT true for all $\mathbf{u}, \mathbf{v}$. For instance, if I take $\mathbf{u} = blah$ and $\mathbf{v} = ...$ then...
- Remember that in this problem, vectors are polynomials!
- For the function that IS linear, you do not need to prove it is linear.
- For the function that is linear, I want you to begin by writing down where an arbitrary input goes. An arbitrary input will be:
$$p(t) = at^3 + bt^2 + ct + d.$$
You should calculate what your transformation $T(p(t))$ is terms of a, b, c, d. You can use this to help you determine what the **kernel** and **range** of the transformation are.
- The best answer will be to explicitly give your kernel and range as the span of a set of vectors. Again: vectors are polynomials in this problem.

By the way, I know how much work you are all doing this semester and I know how hard it is with all the different assignments and readings you have in all your classes. I am SO impressed with you for all your hard work and dedication. I know these assignments can be long, but every problem I assign, I do it for a reason - maybe it helps give you practice, or shows you another side of something, or maybe it helps you practice your proofs. We learn math by solving problems and I want to help you do that by hopefully picking some interesting problems. This class will pick up a bit in the final weeks but hang in. Remember my main goal is for us to learn and we learn through pushing ourselves with challenges.

6 Written Homework 6

Due: Friday November 4th

1. Here's a new example of a **vector space**. The set of all 2×3 matrices forms a vector space. This makes sense because given any two matrices of the same size, we can add them. We can scale a matrix by multiplying all of its entries by the scalar. The zero vector is the matrix consisting of all zeros.

   (a) As a warmup, write down three elements in $\text{Span}\{\begin{bmatrix}1 & 1 & 2\\ 0 & 0 & 4\end{bmatrix}\}$.

   (b) Now write down the **general form** of a vector in

   $$\text{Span}\{\begin{bmatrix}1 & 1 & 2\\ 0 & 0 & 4\end{bmatrix}, \begin{bmatrix}0 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}, \}.$$

   As a start, write down a general linear combination using weights c_1 and c_2. Then combine your vector into a single matrix.

   (c) There is nothing special about 2×2 matrices. The set of all 2×2 matrices forms a vector space as well. In this space, prove that the set of matrices below is linearly independent:

   $$\{\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\}.$$

   Hint: do this by writing really clearly: "Suppose that $c_1\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix} + c_2\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} + c_3\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$" and then set up a system of equations based on the fact that you want all 4 entries to be zero. deduce that the three coefficients must all be zero. This is the way we write a linear independence proof for general vector spaces.

   (d) Let $M_{m\times n}$ be the vector space of all $m\times n$ matrices. Write down the simplest basis you can for the vector space $M_{4\times2}$. Hint: Your basis should have 8 vectors and if you choose wisely, each vector will only have one nonzero entry. You do not need to prove that your set is a basis.

   (e) Which of the following are subspaces of $M_{2\times2}$? No proofs here, just think through the properties.

   (a) $\{\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix} \mid a, b \in \mathbb{R}\}$

   (b) $\{\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix} \mid a + b = 0\}$

   (c) $\{\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix} \mid a + b = 5\}$

   (d) $\{\begin{pmatrix}a & c\\ 0 & b\end{pmatrix} \mid a + b = 0, c \in \mathbb{R}\}$

   (Answers in a footnote)‡

‡yes, yes, no, yes

2. For each of the following, explain why it is NOT a subspace of the given vector space. Your explanation should be one of the following:

- It doesn't contain the 0 vector and here's why .....
- It is not closed under addition because these vectors *blah* and *blah* are in H but their sum is NOT.
- It is not closed under scalar multiplication because these vector *blah* is in H, but the multiple $c \cdot blah$ is not in H.
- Naturally you will not write blah, but you will give specific vectors.

   (a) The set H of 2×2 matrices of determinant ≥ 0 is not a subspace of $M_{2\times2}$.

   (b) The set

$$H = \{ \text{ all matrices of the form } \begin{bmatrix} a & 1 \\ b & c \end{bmatrix} \}$$

   is not a subspace of $M_{2\times2}$.

   (c) The set of all invertible 3×3 matrices is NOT a subspace of $M_{3\times3}$.

3. In this problem you will write a **proof** of the following statement: "Let A be an 3×3 matrix and define the following set H:

$$H = \{v \in \mathbb{R}^3 \ : \ Av = 5v\}.$$

   Prove that H is a subspace of $\mathbb{R}^3$.

   Check out the video in the video folder.

4. To follow up on the previous problem. You showed that the set

$$H = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \ : \ A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 5 \begin{bmatrix} x \\ y \\ z \end{bmatrix} \}$$

   is always a subspace of $\mathbb{R}^3$. But what the heck is it? It is a line? A plane? Maybe is it all of $\mathbb{R}^3$?

   There are lots of matrices below and I want you to calculate (groan - but it's good for our health) the solution spaces to these systems of equations.

   To be clear: You will solve the equation $A\mathbf{v} = 5\mathbf{v}$ for all the matrices below.

   (a) For each problem, write the subspace H as the span of a set of vectors.

   (b) Can you describe your subspace in words. Is it the y axis? Is it the yz plane? Something different?

   (c) I know drawing pictures is hard, but if you can try and draw your subspace this will help.

   Your process should involve solving a **homogeneous system**, that is, one with 0's on the right.

   I promise you will see the payoff for this once we get to Chapter 5. To tell you what to expect, you should get as answers (in some order) [All of $\mathbb{R}^3$, the x_3-axis, only the zero vector, the x_1, x_2-plane.]

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 5 & 0 & 2 \\ 0 & 5 & 2 \\ 0 & 0 & 7 \end{bmatrix}, \quad \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad \begin{bmatrix} 6 & 0 & 0 \\ 1 & 6 & 0 \\ 1 & 1 & 5 \end{bmatrix}$$

$$A = \begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 6 & 9 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

5. Above is a matrix and its echelon form. Use this information to find the null space and column space for A. Clearly indicate the dimension of each space. **The dimension of a subspace is defined to be the number of elements in a basis for H.**

6. Suppose that H is a subspace of $\mathbb{R}^9$ and that a basis for H has 6 vectors. What is the dimension of H?

7. Explain why the following vectors do NOT form a basis for $\mathbb{R}^3$:

$$\begin{bmatrix} 1 \\ -6 \\ -7 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ 7 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 8 \\ 9 \end{bmatrix}$$

(Try to give an answer that does not rely on a calculation. Think about about what you can say about these vectors generally.)

8. Find the vector $\vec{x}$ determined by the given coordinate vector $[\vec{x}]_B$ and the given basis B.

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}, [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

9. Suppose that H is a subspace of $M_{2,3}$ and H has basis:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -1 & -1 \\ 0 & 7 & -4 \end{bmatrix} \right\}$$

Then find the vector $\mathbf{x}$ whose $\mathcal{B}$ coordinates are $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

10. Suppose that H is a subspace of P_4 and H has basis:

$$\mathcal{B} = \{1, 1+t, 1+t+t^2, 1+t+t^2+t^3\}$$

What is the dimension of H? Find the vector $\mathbf{x}$ whose $\mathcal{B}$ coordinates are $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$.

11. In the previous problem, find the coordinates of the vector t^2, then find the coordinates of t. (Try to do these mentally - can you see how to combine these basis vectors to get these vectors?) Now find the coordinates for $t^2 + t$. Did you get that “the coordinates for $t^2 + t$” = “the coordinates for t^2 + the coordinates for t”? (You should!)

7 Written Homework 7

Due: Friday November 11th

The rank-nullity theorem says that: If $T : V \to W$ is a linear transformation then

$$\dim \ker T + \dim \text{range} T = \dim V.$$

1. Some short applications of the Rank Nullity Theorem
   (a) If the subspace of all solutions to $A\vec{x} = \vec{0}$ has a basis consisting of three vectors and if A is a 5×7 matrix, what is the rank of A?
   (b) What is the rank of a 4×5 matrix whose null space is 3 dimensional?
   (c) If the rank of a 7×6 matrix is 4 what is the dimension of the solution space of $A\vec{x} = 0$.
   (d) If possible construct a 3×4 matrix A such that dim Nul(A) = 2 and dim Col(A) = 2.
   (e) Construct a 4×3 matrix with all nonzero entries with rank 1.
2. Some more practice. To help you out, here are some dimension formulas:

   $\dim \mathbb{R}^n = n$, $\quad \dim P_n = n + 1$, $\quad \dim M_{m\times n} = mn$.

   (a) Let's get some practice. The formula above says that $\dim P_4 = 5$. Find a basis of P_4 and see that it has 5 elements. (Answer in footnote[§])
   (b) Check that $\dim M_{2\times 3} = 6$ by finding a basis that has 6 elements. Hint: choose your matrices to have all zeros except for one 1.
   (c) If you have a linear transformation $T : P_9 \to M_{2\times 2}$ and the dimension of the range of T is equal to 3 what would the dimension of $\ker T$ be? (This is just asking you to apply the rank nullity theorem. At the top of this page. Answer: 7)
   (d) If you have a linear transformation $T : M_{2\times 4} \to \mathbb{R}^6$ whose kernel is 4 dimensional, what is the dimension of the range of T? (Answer: 4)
   (e) If you have a linear transformation $T : V \to W$ where $\dim V = 13$ and $\dim W = 27$ and $\ker T$ has basis consisting of 3 vectors, then what is the dimension of the range of T?
3. The following are all linear transformations between vector spaces.

   For each problem:

   I.) Fill in the blank based on what the rank nullity theorem says:

   $$\dim \ker T + \dim \text{range} T = \underline{\qquad}$$

   (fill in the appropriate dimension.) Remember that the rank nullity theorem says that this sum is equal to the dimension of the **domain** of T. So you'll want to figure out that dimension and put it there.)

   II.) Find a basis for $\ker T$

   III.) Find a basis for range T

   IV.) Check that they satisfy $\dim \ker T + \dim \text{range} T = \dim$ **domain**. If you feel stuck, just try to plug in some random inputs to get a feel for what's going on.

[§] $\{1, t, t^2, t^3, t^4\}$ is a basis.

Make sure for each of the problems below you complete ALL of parts I) - IV)

(a) $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(\vec{x}) = A\vec{x}$ where

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 1 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

(In this case, $\ker T$ is just $\text{nul}(A)$ and $\text{range}T = \text{col}(A)$. This example is review - go back and check how to find a basis for the null and column space of a matrix.)

(b) $T : P_4 \to P_4$ given by $T(p(t)) = p''(t)$.

(c) $T : M_{2\times 3} \to M_{3\times 3}$ given by

$$T(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}) = \begin{bmatrix} a & 0 & b \\ 0 & b & 0 \\ b & 0 & -a \end{bmatrix}.$$

(d) Let $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$ Define the transformation

$$T : P_2 \to M_{2\times 2}$$

$$T(at^2 + bt + c) = aA^2 + bA + cI.$$

Where I is the 2×2 identity matrix. (This one is fun!)

4. For the following two problems you are given two bases $B = \{\vec{b}_1, \vec{b}_2\}$ and $C = \{\vec{c}_1, \vec{c}_2\}$ for $\mathbb{R}^2$. Write down the "change of coordinate" matrices going both from B to C and also from C to B. These are also sometimes called the "change of basis" matrices. Label them like we did in class / book as $P_{C\leftarrow B}$ and $P_{B\leftarrow C}$.

The answers to these problems are on the next page. Remember that this homework is to help give you practice - as long as you turn it in you'll get full points. Here I want you to practice following the steps we did in class to see if you get the right answer. You can use a calculator to find the inverses (e.g. on symbolab) but try doing a few yourself by hand to get some practice. These are all reasonable questions that I could ask on the quiz.

(a)

$$\mathbf{b}_1 = \begin{bmatrix} 7 \\ 5 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -3 \\ -1 \end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

(b)

$$\mathbf{b}_1 = \begin{bmatrix} -6 \\ -1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} 6 \\ -2 \end{bmatrix}.$$

(c) In P_2 consider the basis $B = \{1 - 2t + t^2, 3 - 5t + 4t^2, 2t + 3t^2\}$ and the standard basis $C = \{1, t, t^2\}$. Find the change of basis matrices $P_{C\leftarrow B}$ and $P_{B\leftarrow C}$. Use these matrices to find the B coordinates for the polynomial $1 - 2t$ and also the polynomial t^2 Then check your answers by seeing if these coordinates, when used as coefficients in a linear combination indeed give you the right polynomials.

(d) Suppose you have a vector space B with two different bases B and C. If a change of coordinates matrix is given by

$$P_{B\leftarrow C} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

Use this information to answer the following questions:

a If a vector $v \in V$ has coordinates $[v]_C = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ then what is $[v]_B$?

b If a vector $w \in V$ has coordinates $[w]_B = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ then what is $[w]_C$?

Hint: For one of these questions you might want to find the inverse of this matrix, which would be $P_{C \leftarrow B}$.

answers:

a. $P_{C \leftarrow B} = \begin{pmatrix} -3 & 1 \\ -5 & 2 \end{pmatrix}$

$P_{B \leftarrow C} = \begin{pmatrix} -2 & 1 \\ -5 & 3 \end{pmatrix}$

b. $P_{C \leftarrow B} = \begin{pmatrix} 9 & -2 \\ -4 & 1 \end{pmatrix}$

$P_{B \leftarrow C} = \begin{pmatrix} 1 & 2 \\ 4 & 9 \end{pmatrix}$

c.

$P_{B \leftarrow C} = \begin{pmatrix} -23 & -9 & 6 \\ 8 & 3 & -2 \\ -3 & -1 & 1 \end{pmatrix}$

$P_{C \leftarrow B} = \begin{pmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{pmatrix}$

$[1 - 2t]_B = \begin{bmatrix} -5 \\ 2 \\ -1 \end{bmatrix}$

$[t^2]_B = \begin{bmatrix} 6 \\ -2 \\ 1 \end{bmatrix}$

d) To convert from C to B we just use the matrix $P_{B \leftarrow C}$, so multiply by this matrix to get: $[v]_B = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

To go from B to C we need $P_{C \leftarrow B}$ which is the inverse of $P_{B \leftarrow C}$, which is $\begin{bmatrix} 3 & -2 \\ -5 & 3 \end{bmatrix}$. So we get $[w]_C = \begin{bmatrix} 10 \\ -16 \end{bmatrix}$.

8 Written Homework 8

Due: Monday November 21st

1. Suppose you have a vector space B with two different bases B and C. If a change of coordinates matrix is given by

$$P_{B\leftarrow C} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

Use this information to answer the following questions:

a If a vector $v \in V$ has coordinates $[v]_C = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ then what is $[v]_B$?

b If a vector $w \in V$ has coordinates $[w]_B = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ then what is $[w]_C$?

Hint: For one of these questions you might want to find the inverse of this matrix, which would be $P_{C\leftarrow B}$.

2. (Optional - There's not much to write up, but this is a good review question!) Below are a bunch of true statements. Some are about vector spaces/ linear transformations and some are about $\mathbb{R}^n$ / matrices. Your job is to connect the analogous statements. Not every statement will have a match. In some of these, you will have to use your best judgment to decide what the best analogy is.

(a) If $\dim V = r$ then any set of more than r vectors must be linearly dependent

(b) If $T : V \to W$ is a linear transformation then the range of T is a subspace of W

(c) The Column space of an $m \times n$ matrix is a subspace of $\mathbb{R}^m$.

(d) If $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_p}\}$ is a set of vectors in a p-dimensional vector space V and this set spans all of V then this set is linearly independent.

(e) Every basis of an n dimensional vector space will have n elements.

(f) If A is an $m \times n$ matrix that doesn't have a pivot in every row, then the columns of A do not span all of $\mathbb{R}^m$.

(g) If a matrix has more columns than rows then the columns must be linearly dependent

(h) A set of vectors in $\mathbb{R}^n$ is linearly dependent if there is a non-trivial linear combination of them that gives the zero vector.

(i) If the column vectors of a square matrix A has a pivot in every column, then it must have a pivot in every row.

(j) If the column vectors of a square matrix A has a pivot in every row, then it must have a pivot in every column.

(k) The Null space of an $m \times n$ matrix is a subspace of $\mathbb{R}^n$

3. Suppose that $\mathbf{w}$ is a vector that is orthogonal (i.e. perpendicular) to the vectors $\mathbf{u}, \mathbf{v}$. (Recall this means that $\mathbf{w} \cdot \mathbf{u} = 0$ and $\mathbf{w} \cdot \mathbf{v} = 0$. Prove that $\mathbf{w}$ is actually perpendicular to **every** vector in the span of $\{\mathbf{u}, \mathbf{v}\}$.

Hint: Start your proof by saying: "Let $\mathbf{z}$ be a vector in the span of $\mathbf{u}, \mathbf{v}$. This means that .... [and remember your goal is to prove that $\mathbf{w}$ is perpendicular to $\mathbf{z}$.

4. Let W be a subspace of $\mathbb{R}^n$. We define a new set, called the "Torero set of W", denoted $\tau(W)$ as follows:

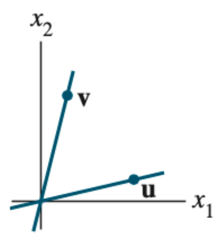
$$\tau(W) = \{v \in \mathbb{R}^n \mid v \cdot w = 0 \text{ for all } w \text{ in } W\}.$$

a) Show that the "Torero set of W" is also a subspace of W. Your proof should clearly check all three of the properties of being a subspace.

b) Prove that if a vector $\mathbf{x}$ is in W and is also in $\tau(W)$ then $\mathbf{x}$ must be the zero vector.

5.

Let u and v be the vectors shown in the figure, and suppose u and v are eigenvectors of a 2×2 matrix A that correspond to eigenvalues 2 and 3, respectively. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$ for each $\mathbf{x}$ in $\mathbb{R}^2$, and let $\mathbf{w} = \mathbf{u} + \mathbf{v}$. Make a copy of the figure, and on the same coordinate system, carefully plot the vectors $T(\mathbf{u})$, $T(\mathbf{v})$, and $T(\mathbf{w})$.



6. Suppose that λ is an eigenvalue of a matrix A. Prove that λ^2 is an eigenvalue of A^2. Hint: Start by saying "I know there is an eigenvector $\mathbf{v}$ with the property that ..."
7. Suppose that λ is an eigenvalue of an invertible matrix A. Show that λ^{-1} is an eigenvalue of A^{-1}.
8. Show that if A^2 is the zero matrix then the only eigenvalue of A is zero. (Hint: Use Exercise 6)
9. Is it possible that A can have two different eigenvalues, but A^2 only has 1? (Hint: Look at diagonal matrices)
10. Is it possible that A can eigenvalues but A^2 has no eigenvalues? (Write your answer clearly.)

Part III

Quizzes

Quiz 1

1. Consider the matrix A:

$$\begin{bmatrix} 2 & 3 & -1 & 7 & 6 \\ 0 & 3 & 3 & -1 & 0 \\ 0 & 1 & 2 & 0 & 4 \end{bmatrix}$$

   (a) (2 points) Explain why the matrix is NOT in Echelon form.

   (b) (2 points) By using row operations, put the matrix in Echelon form. Circle the pivot positions.

$$\begin{bmatrix} 2 & 3 & -1 & 7 & 6 \\ 0 & 3 & 3 & -1 & 0 \\ 0 & 1 & 2 & 0 & 4 \end{bmatrix}$$

2. (2 points) Write down a 4×6 matrix in echelon form that has pivots in columns 1, 3, 4.

Turn the page over for the last problem:

3. (6 points) A student is solving a linear system. She writes down the augmented matrix for the system and uses row operations to put the matrix in echelon form and gets the matrix below:

$$\begin{bmatrix} 1 & -2 & 0 & 7 & -6 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) How many variables are in the system?

(b) How many equations are in the system?

(c) x_1 is a [basic variable/free variable] (circle one)

(d) x_2 is a [basic variable/free variable] (circle one)

(e) If the system is consistent, write down the solution like we did in class. If it is inconsistent, explain why.

Quiz 2 **Name:** ________________________________

1. The augmented matrix for a consistent system of 8 equations in 4 variables can have at most ______ pivots.

2. Consider the 3 systems below. Circle all systems that are homogeneous. There may be more than one.

$$\begin{cases} x+y-z &= 8 \\ x-y+z &= 0 \end{cases} \qquad \begin{cases} x+7 &= 0 \\ x-y-w+6 &= 0 \\ x-y-z &= 0 \end{cases} \qquad \begin{cases} x+y &= z+w \\ x-6z &= w \\ -x+6w &= y+z \end{cases}$$

3. The zero vector in $\mathbb{R}^2$ can be written as a linear combination of the two vectors $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\0\end{bmatrix}$. (True / False)

4. (Circle or fill in the blanks) A given set of vectors in $\mathbb{R}^m$ is said to **span** all of $\mathbb{R}^m$ if [some / every] vector in $\mathbb{R}^m$ can be written as a linear combination of the given vectors.

5. A student wants to tell if a set of 6 vectors span $\mathbb{R}^5$. The student writes the 6 vectors as columns of a matrix A and row-reduces the matrix to Echelon form. The student gets the matrix below:

$$\begin{bmatrix} 1 & 0 & 0 & 5 & 5 & 1 \\ 0 & 2 & 0 & 5 & 4 & 1 \\ 0 & 0 & 1 & 5 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Which of the following is the best conclusion:

(a) The 6 vectors span $\mathbb{R}^5$ because 6 is greater than 5.

(b) The 6 vectors span $\mathbb{R}^5$ because the linear system is consistent.

(c) The 6 vectors span $\mathbb{R}^5$ because there are free variables.

(d) The 6 vectors do not span $\mathbb{R}^5$ because there is not a pivot in the bottom row.

(e) The 6 vectors do not span $\mathbb{R}^5$ because the system is inconsistent.

6. Let A be the coefficient matrix for a linear system, and let B be the augmented matrix for the system.
   - If the system is consistent then A has [the same / more / less] pivots than B. (Circle one)

7.

For this question only, please write a brief explanation for parts b) and c)

A 4×5 matrix A has three pivots. Choose the best answer for each of the following

(a) In the system $A\mathbf{x} = \mathbf{b}$, $\mathbf{x}$ is a vector in [$\mathbb{R}^4/\mathbb{R}^5/\mathbb{R}^6$] and $\mathbf{b}$ is a vector in [$\mathbb{R}^4/\mathbb{R}^5/\mathbb{R}^6$]

(b) The homogenous system $A\mathbf{x} = \mathbf{0}$ has [0 solutions / 1 solution / infinitely many solutions / cannot tell from the given information.]

(c) The homogenous system $A\mathbf{x} = \mathbf{b}$ has a solution for [some vectors $\mathbf{b}$ / all vectors $\mathbf{b}$ / no vectors $\mathbf{b}$]

Quiz 4 **Name:** ________________________________

Good luck!

1. (5 points) Complete the following definition:

   We say that the vectors $\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}, \mathbf{w_4}$ are linearly **dependent** if

2. (4 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation defined by multiplication by the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 1 & 1 & -1 \\ 4 & 1 & 0 \end{bmatrix}$$

   You can use that A has 3 pivots

   (a) What is the domain of T?

   (b) What is the co-domain of T?

   (c) Is T 1-1? [Yes/No]

   (d) Is the range of T equal to $\mathbb{R}^4$? [Yes/No]

3. (2 points) A student is writing a proof about a linear transformation T and writes the following:

   "Because ________________________________, and

$$c_1 T(\mathbf{v}) + c_2 T(\mathbf{w}) = 0$$

   we can conclude that c_1 and c_2 are both zero."

   Which of the following could go into the blank to make this a valid explanation: (Only one is correct)

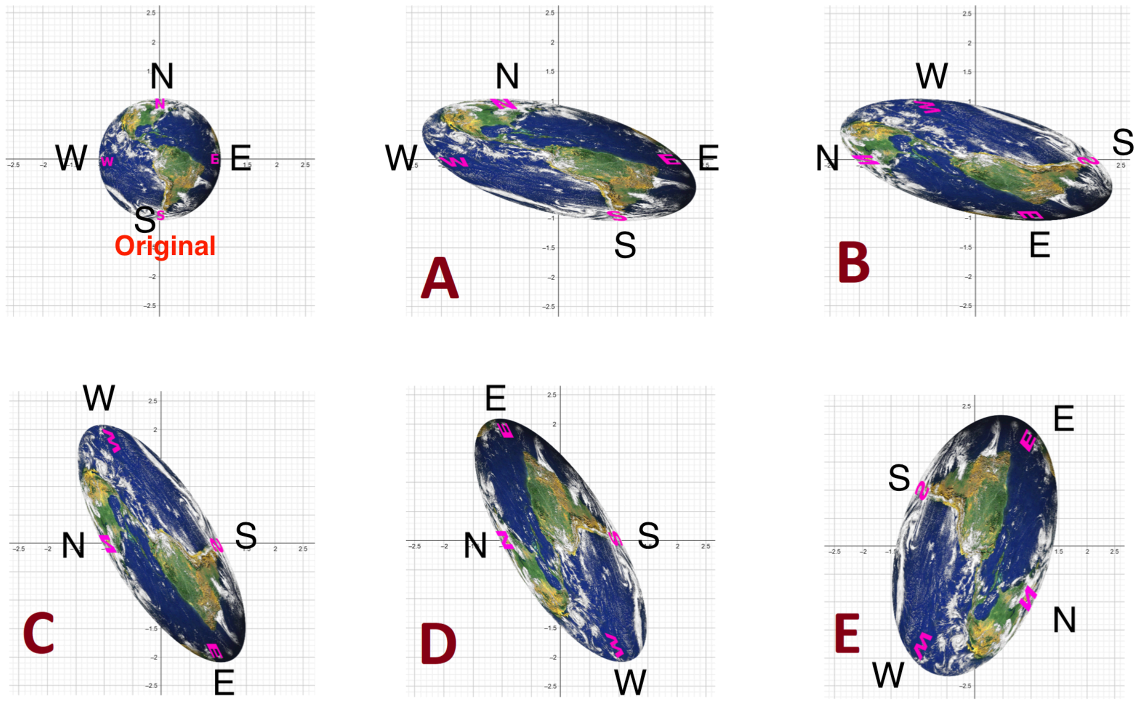
   (a) $\{\mathbf{v}, \mathbf{w}\}$ is a linearly dependent set

   (b) $\{\mathbf{v}, \mathbf{w}\}$ is a linearly independent set

   (c) $\{T(\mathbf{v}), T(\mathbf{w})\}$ is a linearly dependent set

   (d) $\{T(\mathbf{v}), T(\mathbf{w})\}$ is a linearly independent set

4. (2 points) A student applies the linear transformation given by the matrix $\begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}$. Select which image below best represents the effect of this transformation on the original image.



5. (2 points) Write down the standard matrix for the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that first reflects across the x axis and then across the y axis.

Quiz 5 **Name:** ______________________________

Was takehome - homework

Quiz 6 **Name:** ______________________________

1. Determinant Questions

   (a) If $\det(A) = 0$ then A [is/is not] invertible. If $\det(A) \neq 0$ then A [is/is not] invertible.

   (b) A student wants to find the volume of the 3D parallelepiped spanned by three vectors. The student puts these vectors into a matrix and takes the determinant and gets -12. This means the volume of the parallelepiped is ____________.

   (c) Suppose that A is a 4×4 matrix. President Harris performs the following row operations:

   - Replace row 1 with $R_1 + 2R_3$
   - Replace row 2 with $3R_2$

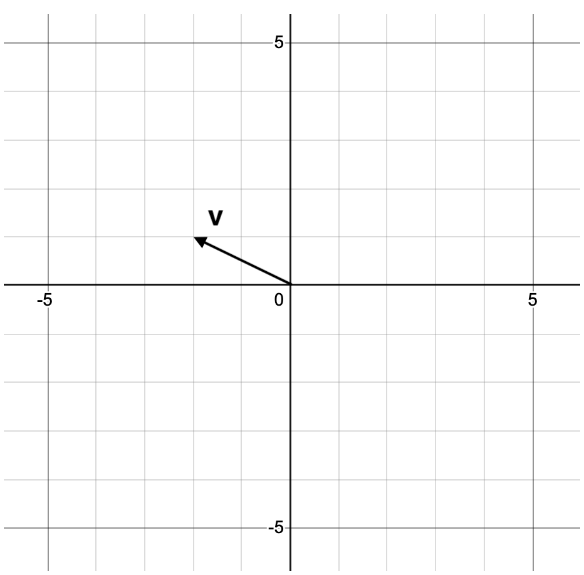
   He then ends up with a matrix B that has determinant 30. What is the determinant of A?

   (d) Explain why you can easily see that the following matrix has determinant zero:

$$\begin{bmatrix} 2 & 3 & 4 & 5 & 5 \\ 102 & 8 & 44 & 0 & 0 \\ 345 & 7 & 404 & 1 & 1 \\ 422 & 6 & 404 & 1 & 1 \\ 57 & 5 & 404 & 1 & 1 \end{bmatrix}$$

   Hint: Don't even think aout trying to expand anything out. Think about properties of matrices that would tell you if the matrix had determinant zero. **Make sure your explanation is complete.**

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by multiplication by the matrix $\begin{bmatrix} 4 & 3 \\ 1 & -2 \end{bmatrix}$. Let $\mathbf{v}$ be the vector in the picture. Calculate $T(\mathbf{v})$ and circle your answer. Then draw $T(\mathbf{v})$ in the picture.



3. Complete the following definition:

   Let V be a vector space. We say that H is a subspace of V if H is a _____________ of V and

   1)

   2) H is closed under vector addition.

   3)

4. Which of the following is meant by the property "H is closed under vector addition."

   (a) If you add two elements of V you get an element of H

   (b) If you add two elements of V you get an element of V

   (c) If you add two elements of H you get an element of H

   (d) If you add two elements of H you get an element of V.

5. Let V be the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$. Please mark the following True or False:

   (a) $V = \text{Span}\{\sin x, \cos x\}$

   (b) $\{ae^x + bx^2 \ : \ a, b \in \mathbb{R}\} = \text{Span}\{e^x + x^2\}$

   (c) $\{ae^x + bx^2 \ : \ a, b \in \mathbb{R}\} = \text{Span}\{e^x, x^2\}$

   (d) The set $\{1, x, x + 1\}$ is linearly independent

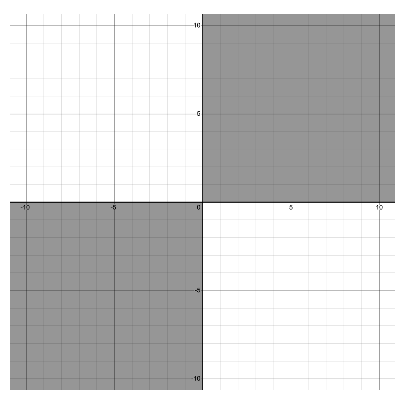
   (e) $\{1, x\}$ is a subspace of V.

   (f) $\text{Span}\{1, x\}$ is a subspace of V

   (g) $\text{Span}\{e^x + x^2\}$ is a subspace of V

6. Write down three different non-zero vectors in $\text{Span}\{\sin x + \cos x, e^x\}$.

7. Consider the shaded region H, which consists of all vectors $\begin{bmatrix} a \\ b \end{bmatrix}$ in $\mathbb{R}^2$ such that $ab \geq 0$. Determine whether H is a subspace. Explain your answer.



Quiz 7 **Name:** ______________________________

1. (2 points) Fill in the blank. If a 4×5 matrix has 2 free variables, then the rank of the matrix is ____________.

2. (3 points) Complete the following definition:

   If A is an $m \times n$ matrix, then the column space of A, denoted $\text{Col}(A)$ is

3. (6 points) Below is a matrix A and its reduced Echelon form:

$$A = \begin{pmatrix} 1 & 2 & 5 & -2 & 3 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 3 & -1 & 2 \end{pmatrix}, \text{ and the RREF is } \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

   (a) Write down a basis for $\text{Col}(A)$

   (b) Write down a basis for $\text{Nul}(A)$

4. You are working in the vector space V of all functions and discover a subspace H. You work out that the following set is a basis for H.

$$\mathcal{B} = \{e^x, \sin x + 2\cos x, x^7\}.$$

   (a) (4 points) What is the dimension of H? Explain your answer.

   (b) (3 points) You want to find the vector $\mathbf{v}$ and know its coordinates are $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$. What is $\mathbf{v}$?

   (c) (3 points) Write down the $\mathcal{B}$-coordinates for the vector $\mathbf{w} = 8x^7$.

$$[\mathbf{w}]_{\mathcal{B}} =$$

5. Please explain your reasoning in a sentence or two in the following problems:

(a) (5 points) A student is studying a linear transformation $T : P_5 \to \mathbb{R}^8$ and wants to write down a basis for $\ker T$. The student writes down the set $\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\}$. Explain why the student's answer must be incorrect.

(b) (5 points) A student is studying a 5×2 matrix A and says that Nul(A) is 3 dimensional. Explain why the student must have made a mistake in their calculation.

(c) (5 points) If a linear transformation $T : P_5 \to M_{2\times 4}$ has a 2 dimensional kernel, then what is the dimension of its range? Briefly justify your answer.

Quiz 8

Please write your answers on a separate sheet of paper.

1. If $\mathbf{v}$ is an eigenvector of A with $\lambda = 2$, what is $A^4(5\mathbf{v})$. Show your work in simplifying.

2. A. Calculate the eigenvalues for the matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$. Show your work.

   B. Now find the eigenvectors for each of these eigenvalues and draw a picture of the two eigenspaces like we've been doing in class. Do your row operations by hand.

   C. Does $\mathbb{R}^2$ have a basis consisting of eigenvectors of A? If so, write one down. If not, explain why.

3. Let $B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

   A. What are the eigenvalues for B?

   B. Find the eigenvectors for each of these eigenvalues and draw a picture of the eigenspaces.

   C. Does $\mathbb{R}^3$ have a basis consisting of eigenvectors of B?

4. Answer the following true/false questions (no explanation needed)

   1.) If $\mathbf{v}$ is a nonzero vector and $A\mathbf{v} = 5\mathbf{v}$ then 5 is an eigenvalue of A.

   2.) If $\mathbf{v}$ is a nonzero vector and $A\mathbf{v} = 5\mathbf{v}$ then $\mathbf{v}$ is an eigenvector of A.

   3.) If $\mathbf{v}$ is a nonzero vector and $A\mathbf{v} = 5\mathbf{v}$ then $3\mathbf{v}$ is an eigenvector of A.

   4.) If A is a lower triangular matrix with 0s on the diagonals then A has only one eigenvalue.

   5.) If $\lambda = 0$ is an eigenvalue for a matrix A then A is invertible.

   6.) A set of eigenvectors is always linearly independent.

   7.) A set of eigenvectors, each with distinct eigenvalues is linearly independent.

   8.) Let λ be an eigenvalue of an $n \times n$ matrix. The set of eigenvectors with eigenvalue λ together with the zero vector forms a subspace of $\mathbb{R}^n$.

   9.) Let A be an $n \times n$ matrix. The set of all eigenvectors together with the zero vector forms a subspace of $\mathbb{R}^n$.

   10.) The zero matrix has no eigenvalues.

5. You are studying a matrix A and notice that the matrix $A - 7I$ has RREF

$$\begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   The above information tells you some (partial) information about the eigenvalues and eigenvectors of A. Describe as much as you can about the matrix A and its eigenvalues.

   A correct answer will include phrases like: "I know that BLAH is an eigenvalue because" "The eigenspace corresponding to BLAH is BLAH dimensional and has basis BLAH."

   Warning: The above is the RREF for $A - 7I$. It will NOT be possible to figure out what the original matrix A is from this. If you are stuck, go back and look at how you solved 3.

Part IV

Midterm and Final

1 Midterm

Linear Algebra Midterm

Your Name

- You have 55 minutes to do this exam.
- No calculators or notes are allowed.
- For justifications, please use complete sentences and make sure to explain any steps which are questionable.
- If needed, you may use:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \text{ provided, } ad - bc \neq 0.$$

- Good luck!

Problem	Total Points	Score
1 - Definitions	12	
2 - Calculations	22	
3 - Writing down examples	12	
4 - Short Answer, Multiple Choice, and True False	30	
5- Explanations ("Proofs")	24	
Total	100	

GOOD LUCK!!!!

1. (12 points) Complete the following definitions:

   (a) The set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent if

   (b) The vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots \mathbf{v}_p$ are linearly independent if

   (c) An $n \times n$ matrix A is called invertible if

2. (20 points) Calculations

   (a) (5 points) A student is solving an equation:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

   By carefully doing the arithmetic described above, you should arrive at a system of linear equations. Please write down the augmented matrix for this system, showing your work. Just do this slowly and carefully and you'll do great!

(b) (4 points) Calculate: $\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

(c) (3 points) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^4$ is defined by $\mathbf{x} \mapsto A\mathbf{x}$ where A is the matrix $\begin{bmatrix} 1 & -1 \\ -1 & 2 \\ 2 & 3 \\ 1 & 3 \end{bmatrix}$.

Write down three different possible input vectors. Do not calculate the outputs.

(d) (4 points) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ has the following properties.

$$T(\mathbf{e}_1) = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$$

$$T(\mathbf{e}_1 + 2\mathbf{e}_2) = \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix}$$

Write down the **standard matrix** for T. (Partial credit will be given - please don't spend too much time on this one - you can come back later)

(e) (6 points) A student is solving a linear system. She writes down the augmented matrix for the system and uses row operations to put the matrix in echelon form and gets the matrix below:

$$\begin{bmatrix} 1 & -2 & 0 & 7 & -6 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

i. How many variables are in the system?

ii. x_1 is a [basic variable/free variable] (circle one)

iii. x_2 is a [basic variable/free variable] (circle one)

iv. The system is [consistent/inconsistent] because ______________________________.

3. (12 points) Writing Examples

(a) (4 points) Write down a 3×2 matrix A in Echelon form so that the equation $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

(b) (5 points) Write down three different vectors in $\mathbb{R}^4$ that are in Span$\{\mathbf{e}_1 + \mathbf{e_2}, \mathbf{e_3}\}$. Please write your answers as column vectors.

(c) (3 points) Write down a 2×2 matrix E such that E is NOT invertible. You do NOT need to justify why E is not invertible.

4. (30 points - all are 1 point unless otherwise noted) Short Answer and True False

1.) If A and B are $n \times n$ matrices then AB is [sometimes/always/never] equal to BA.

2.) If A is a 7×5 matrix, then multiplication by A defines a linear transformation with domain ____________.

3.) Let G be a matrix and let $\mathbf{x}, \mathbf{y}$ be vectors so that $G\mathbf{x} = \mathbf{y}$. If $\mathbf{y}$ has 8 rows then this means that G has [8 rows / 8 columns]

4.) (3 points) If the augmented matrix for a system of equations is 4×6

then the system has ____________ equations and ____________ variables.

5.) (6 points) The matrix $B = \begin{bmatrix} 1 & -2 & 0 \\ 4 & 5 & 0 \\ -7 & 3 & 0 \end{bmatrix}$ defines a linear transformation T that is [1-1 / not 1-1] and

that is [onto / not onto]. This matrix is [invertible / not invertible].

This matrix has ____________ pivots.

6.) If $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4} \in \mathbb{R}^8$ and $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is a linearly dependent set, then $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$ is [sometimes/always/never] linearly dependent.

7.) Every matrix is row equivalent to one and only one matrix in reduced row echelon form. [True / False]

8.) The span of the columns of the matrix $\begin{bmatrix} 1 & 4 & 5 & 7 \\ 2 & 3 & 0 & 0 \end{bmatrix}$ is a line in $\mathbb{R}^2$ [True / False]

9.) Any set containing the zero vector is linearly dependent [True/False]

10.) (2 points) If the vector $\mathbf{v} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$ is written as a linear combination of the vectors $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, \mathbf{e_4}$ in $\mathbb{R}^4$:

$$\mathbf{v} = c_1\mathbf{e_1} + c_2\mathbf{e_2} + c_3\mathbf{e_3} + c_4\mathbf{e_4}$$

then c_3 = ____________

11.) (2 points) A student wants to tell if a set of 6 vectors span $\mathbb{R}^5$. The student writes the 6 vectors as columns of a matrix A and row-reduces the matrix to Echelon form. The student gets the matrix below:

$$\begin{bmatrix} 1 & 0 & 0 & 5 & 5 & 1 \\ 0 & 2 & 0 & 5 & 4 & 1 \\ 0 & 0 & 1 & 5 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Which of the following is the best conclusion:

i. The 6 vectors span $\mathbb{R}^5$ because 6 is greater than 5.

ii. The 6 vectors span $\mathbb{R}^5$ because the linear system is consistent.

iii. The 6 vectors span $\mathbb{R}^5$ because there are free variables.

iv. The 6 vectors do not span $\mathbb{R}^5$ because there is not a pivot in the bottom row.

v. The 6 vectors do not span $\mathbb{R}^5$ because the system is inconsistent.

12.) (2 points) If A is a 4×5 matrix with 3 pivots and $\mathbf{b}$ is a vector in $\mathbb{R}^4$ then the system $A\mathbf{x} = \mathbf{b}$

i. will have a unique solution regardless of what $\mathbf{b}$ is.

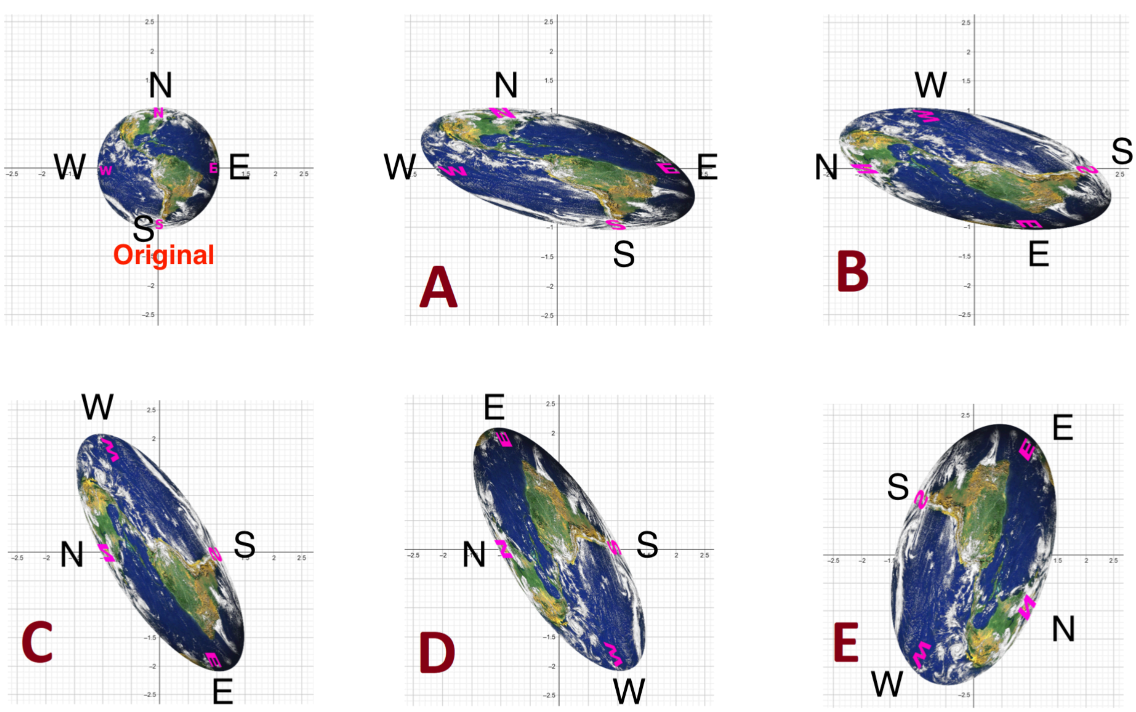
ii. will have infinitely many solutions regardless of what $\mathbf{b}$ is.

iii. will have no solutions regardless of what $\mathbf{b}$ is.

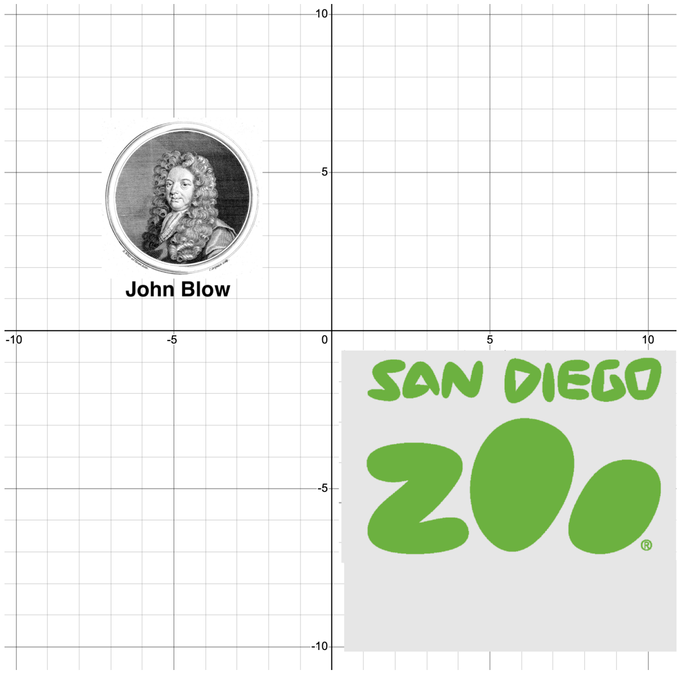
iv. may or may not have a solution, but if it does, the solution is unique.

v. may or may not have a solution, but if it does, there will be infinitely many solutions.

13.) (2 points) A student applies the linear transformation given by the matrix $\begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}$. Select which image below best represents the effect of this transformation on the original image.



14.) (4 points) Baroque organist and composer John Blow would like to visit the San Diego Zoo. Which of the following matrices would define a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that would send him to the zoo? (Circle all that apply)



$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 1 \\ -5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

15.) The matrix $\begin{bmatrix} 2 & 3 \\ 7 & 8 \end{bmatrix}$ is invertible [True/False]

16.) If D, E, F are 3×3 matrices and $DEF = I_3$

then which of the following is the inverse of D: [E / F / DE / EF]

5. (21 points)

> Important: Choose ANY 3 of the five following problems. If you do not clearly indicate which three you want me to grade, I will grade a) b) and c).

(a) (8 points) If A is a 4×3 matrix and B is a 3×7 matrix then AB [sometimes/always/never] has columns that are linearly independent. Explain your reasoning. (Hint: How big is the matrix AB?)

(b) (8 points) Suppose that A is a 4×3 matrix and you are given that $A\begin{bmatrix}1\\2\\3\end{bmatrix} = A\begin{bmatrix}2\\0\\2\end{bmatrix}$. Explain why the columns of A must be linearly dependent. (Hint: Think about what the transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ is doing to these vectors. Feel free to use the tests for different properties that we have learned in class.)

(c) (8 points) Suppose that A is an $n \times n$ invertible matrix. Explain why the columns of A^2 must be linearly independent. (This was one of your homework questions)

(d) (8 points) If $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4} \in \mathbb{R}^3$ and $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is a linearly independent set, then $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$ is [sometimes/always/never] linearly independent. Support your answer.

(e) (8 points) Suppose you are given that

- $T : \mathbb{R}^4 \to \mathbb{R}^8$ is a linear transformation
- $\mathbf{u}, \mathbf{v} \in \mathbb{R}^4$ are linearly independent vectors
- $T(u), T(v)$ are linearly dependent

Determine whether the vector $5u - 3v$ is sometimes/always/never the zero vector. Justify your answer.

2 Final Exam

Name : ______________________________

Linear Algebra Final - Fall 2022

- You have 120 minutes to do this exam.
- No calculators or notes are allowed.
- For justifications, please use complete sentences and make sure to explain any steps which are questionable.
- If needed, you may use:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \text{ provided, } ad-bc \neq 0.$$

For $T : V \to W$,

$$\dim \ker T + \dim \operatorname{range} T = \dim V.$$

- Good luck!

Problem	Total Points	Score
1 - Short Answer	26	
2 - Null space / Column space + Ker / Range	20	
3 - Short Answer (Subspaces/Linear Transformations)	10	
4 - Two Problems where you need to explain	12	
5 - Short Answer (eigenvalues and diagonalization)	14	
6 - Explanations / Calculations (eigenvalues and diagonalization)	18	
Total	100	

1. Some questions about matrices and solving systems of equations:

   (a) (2 points) Complete the following definition: The set of vectors $\{v_1, \ldots, v_n\}$ is linearly independent if

   (b) (1 point) If a 4×5 matrix A has 4 pivots, the equation $A\mathbf{x} = \mathbf{b}$ [sometimes/always/never] has a solution.

   (c) (1 point) If a 4×2 matrix A has 2 pivots, the equation $A\mathbf{x} = \mathbf{b}$ [sometimes/always/never] has a solution.

   (d) (1 point) If a 4×5 matrix A has 3 pivots, the equation $A\mathbf{x} = \mathbf{b}$ [sometimes/always/never] has a solution.

   (Room for scratch work, no explanations necessary)

   (e) (2 points) You solve a system of equations determined from the equation $A\mathbf{x} = \mathbf{0}$ for a 4×3 matrix A with two free variables. Which of the following is the best description of the solution.

      i. A line in $\mathbb{R}^3$
      ii. A line in $\mathbb{R}^4$
      iii. A line in $\mathbb{R}^5$
      iv. A plane in $\mathbb{R}^3$
      v. A plane in $\mathbb{R}^4$
      vi. A plane in $\mathbb{R}^5$

   (f) (1 point) Can the columns of a 7×4 matrix be linearly independent? [Yes / No]

   (g) (1 point) If A is a 3×4 matrix with rank 3 then

      i. The columns of A span all of $\mathbb{R}^4$
      ii. The columns of A span all of $\mathbb{R}^3$
      iii. The columns of A span a 3-dimensional subspace of $\mathbb{R}^4$
      iv. None of the above are correct.

   (h) (1 point) If a 5×10 matrix A has 3 pivots then

      i. The null space of A is 5 dimensional
      ii. The null space of A is 3 dimensional
      iii. The null space of A is 8 dimensional
      iv. The null space of A is 7 dimensional
      v. The null space of A is 2 dimensional
      vi. The null space of A is trivial and consists of only the zero vector.

(i) (4 points) In the vector space V of all functions, a student is working with a basis for a subspace H:

$$\mathcal{B} = \{e^x, \sin x + 2\cos x, x^2\}.$$

What is the dimension of the subspace H? ____________

If $[f]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$, then what function is f? ____

Of the statements below, circle all that are correct statements:

i. $H = \{e^x, \sin x + 2\cos x, x^2\}$

ii. $H = \text{Span}(\{e^x, \sin x + 2\cos x, x^2\})$

iii. $V = \text{Span}(\{e^x, \sin x + 2\cos x, x^2\})$

iv. $\mathcal{B}$ is a subspace of V

v. $\mathcal{B}$ is a subspace of H

(j) (1 point) The columns of an invertible 4×4 matrix will [sometimes/always/never] form a basis for $\mathbb{R}^4$.

(k) (5 points) Suppose that A is a 3×3 matrix whose RREF is $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

A is [invertible/not invertible/cannot tell]

1 is an eigenvalue of A [yes / no / cannot tell]

0 is an eigenvalue of A [yes / no / cannot tell]

The columns of A are linearly independent [yes / no / cannot tell]

The column space of A is equal to $\{\begin{bmatrix} a_1 \\ a_2 \\ 0 \end{bmatrix}$ for $a_1, a_2 \in \mathbb{R}\}$ [yes /no / cannot tell]

(l) (1 point) Suppose a subspace H of $\mathbb{R}^3$ has basis $\{\mathbf{v_1}, \mathbf{v_2}\}$. If $\mathbf{y} \in \mathbb{R}^3$ then $\mathbf{y}$ can [sometimes/always/never] be written as a linear combination of the vectors $\mathbf{v_1}, \mathbf{v_2}$.

(m) (1 point) Suppose a subspace H of $\mathbb{R}^3$ has basis $\{\mathbf{v_1}, \mathbf{v_2}\}$. If $\mathbf{y} \in H$ then $\mathbf{y}$ can [sometimes/always/never] be written as a linear combination of the vectors $\mathbf{v_1}, \mathbf{v_2}$.

(n) (4 points) A student is studying a function $f : \mathbb{R}^2 \to \mathbb{R}^3$ and learns that the equation $f(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has exactly three solutions. Explain why you know that T cannot be a linear transformation.

2. (a) (4 points) For a general $m \times n$ matrix A define Col(A) and Nul(A).

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(b) (8 points) The matrix A and its RREF form B are below, write down a **basis** for Col(A) and a **basis** for Nul(A)

$$A = \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 5 & -10 & 0 & 5 \\ 0 & 5 & -10 & 3 & 8 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Fill in the blanks below:

The rank of A is ______________.

The rank of B is ______________.

The dimension of Col(A) is ______________.

The dimension of Nul(A) is ______________.

(c) (8 points) Suppose that $T : P_3 \to M_{3\times3}$ is the transformation given by

$$T(a + bt + ct^2 + dt^3) = \begin{bmatrix} a-b & 0 & c \\ 0 & c & 0 \\ c & 0 & 0 \end{bmatrix}.$$

Write down a basis for $\ker T$ and for range(T). Please support your answers with clear and careful explanations justifying your steps. You may use the rank-nullity theorem to help with your answer.

Fill in the blanks below:

The dimension of range(T) is ____________.

The dimension of $\ker(T)$ is ____________.

3. Some questions about subspaces and linear transformations

(a) (2 points) Complete the following definition: We say that H is a subspace of a vector space V if H is a subset of V and

(b) (1 point) Span$\{\begin{bmatrix} 4 \\ 2 \end{bmatrix}\}$ is a subspace of $\mathbb{R}^2$. [True / False]

(c) (1 point) If T is a linear transformation from a vector space V to a vector space W then the kernel of T is a

i. subspace of V

ii. subspace of W

iii. A subset of V but not necessarily a subspace of V

iv. A subset of W but not necessarily a subspace of W.

(d) (1 point) If $T : V \to W$ is a linear transformation and you know that $T(3\mathbf{v} + 2\mathbf{w}) = 2\mathbf{u}$ which of the following would be equal to $T(6\mathbf{v} + 4\mathbf{w})$?

A) $\mathbf{u}$

B) $4\mathbf{u}$

C) $3\mathbf{v} + 2\mathbf{w} - 2\mathbf{u}$

D) $\mathbf{v}$

(e) (5 points) Consider the subset H of P_2:

$$H = \{2t^2 + b \ : \ b \in \mathbb{R}\}$$

Circle ALL of the following that are true:

i. H does not contain the zero vector

ii. H is not closed under scalar multiplication

iii. H is not closed under vector addition

iv. H is a subspace of P_2

v. H is a subspace of $\mathbb{R}^8$.

4. (a) (6 points) Let $A = \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$ and consider

$$H = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \ : \ \text{either } x_1 + x_2 = 0 \text{ or } A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$$

Is H a subspace of $\mathbb{R}^2$? Explain your answer clearly.

(b) (6 points) Bob is studying a linear transformation and wants to calculate the dimension of the range of the transformation, $T : P_3 \to M_{2\times 7}$. Bob discovers that the kernel of T is equal to $\text{Span}\{t, t+1, 2t+1\}$. What is the dimension of Range(T)? Explain your answer.

5. Short Answer about Eigenvalues

(a) (2 points) Complete the following definition: An $n \times n$ matrix A is diagonalizable if

(b) (2 points) Let $Q = \begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}$. Is $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ an eigenvector of Q? If so, find the eigenvalue.

(c) (1 point) Let A be an $n \times n$ matrix. Then A always has at least one eigenvector, namely the zero vector. [True / False]

(d) (1 point) Suppose a 3×3 matrix A has eigenvalues $0.4, 0.6, 1.7$.

i. A is diagonalizable

ii. A is not diagonalizable

iii. We cannot tell if A is diagonalizable.

(e) (1 point) Suppose a 4×4 matrix A has eigenvalues $0.4, 0.6, 1.7$.

i. A is diagonalizable

ii. A is not diagonalizable

iii. We cannot tell if A is diagonalizable.

(f) (1 point) An invertible matrix is always diagonalizable [True/False]

(g) (1 point) Suppose A is a square matrix and $\mathbf{v}, \mathbf{w}$ are nonzero vectors such that $A\mathbf{v} = 3\mathbf{v}$ and $A\mathbf{w} = 3\mathbf{w}$ then

i. $\mathbf{v}, \mathbf{w}$ are linearly independent vectors

ii. $\mathbf{v}, \mathbf{w}$ are linearly dependent vectors

iii. We cannot tell

(h) (1 point) Suppose A is a square matrix and $\mathbf{v}, \mathbf{w}$ are two distinct nonzero vectors such that $A\mathbf{v} = 3\mathbf{v}$ and $A\mathbf{w} = 4\mathbf{w}$ then

i. $\mathbf{v}, \mathbf{w}$ are linearly independent vectors

ii. $\mathbf{v}, \mathbf{w}$ are linearly dependent vectors

iii. We cannot tell

(i) (1 point) Suppose that A is a 5×5 matrix that has 3 eigenspaces, each 1 dimensional. Then

i. A is diagonalizable

ii. A is not diagonalizable

iii. We cannot tell if A is diagonalizable from the given information.

(j) (1 point) Suppose that $A = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}$ Then

i. A is diagonalizable because A is diagonal.

ii. A is diagonalizable because it has two distinct eigenvalues.

iii. A is diagonalizable because A is upper triangular.

iv. More than one of the above is true.

(k) (2 point) Suppose that the characteristic polynomial of a diagonalizable matrix A is $(\lambda - 2)^2(\lambda^2 - 1)$. Write down a diagonal matrix D that is similar to A.

6. (a) (2 points) Find the eigenvalues of the matrix $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

(b) (3 points) Find a basis for the eigenspace corresponding to $\lambda = 3$ for the matrix $\begin{bmatrix} 4 & -1 & 1 \\ 2 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

(c) (6 points) You are given that the characteristic polynomial for a matrix A is $(\lambda - 2)^3(\lambda - 3)$

What are the eigenvalues of A? What can you say about the dimensions of the eigenspaces? How many pivots does A have? Explain your answers

(d) (7 points) Suppose that you have a matrix A with the following properties.

i. A is a 4×4 matrix.

ii. $A \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 3 \end{bmatrix}$

iii. There are nontrivial solutions to $A\mathbf{v} + 5\mathbf{v} = \mathbf{0}$

iv. 10 is an eigenvalue of A.

v. $A - 7I$ has 2 free variables.

Explaining your steps, please write a paragraph or two that answers the following questions: What are the eigenvalues of A? What is the characteristic polynomial of A? Is A invertible? What are the dimensions of the eigenspaces of A? Is A diagonalizable? (Hint: You have enough information to answer these questions. Make sure you offer full justification for the dimensions of the eigenspaces.)