Gröbner Bases for Commutative Algebraists
The RTG Workshop at Utah

Adam Boocher

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1 Introduction

Maybe you've heard of Gröbner bases before. They are the computational and theoretical underpinning of computational algebra software. They're built into Mathematica, Maple, Macaulay2, Magma and probably even some algebra software not starting with an M!

If you've heard of them, it's probably in one of the following contexts:

- Want to find a solution to a system of polynomials in $\mathbb{C}^n$? Gröbner bases can be applied using what is called an "elimination order" and can help you reduce this problem (in some cases) to solving polynomials in a single variable. You can think of it as like performing Gaussian

Elimination on a matrix, and then back-substituting to solve the full system. In fact, in the special case that the polynomials are all linear, finding a Gröbner basis is one in the same as Gaussian Elimination!

- Are you interested in the kernel of a ring map? Equivalently, would you like to know the (closure of the) image of a map of projective varieties? E.g. the image of the map $f : \mathbb{P}^1 \to \mathbb{P}^2$, $f([x, y]) = [x^2, xy, y^2]$ is defined by the computing the following intersection which can be solved using Gröbner bases:

$$(a - x^2, b - xy, c - y^2) \cap \mathbb{C}[a, b, c] = (b^2 - ac).$$

- What about computing syzygies or colon ideals? Or intersections? Gröbner bases facilitate all sorts of computations (and more).

- If one is savvy with Gröbner bases then one could, for instance look at the ideal

$$I = (a^2 + b^2 + c^2 + d^2, d^4 + abcd + c^4, a^2 - 2ac + c^2)$$

  and know that is has height 3 and is thus a complete intersection.

- Can you bound the depth, regularity, or projective dimension of R/I? Can you *degenerate* I to a monomial ideal?

- As a modest goal of these notes, I want to survey the major theorems in this area, and show how useful these techniques can be in simplifying problems. For instance, in Exercise 11 we show how Gröbner basis methods can be used to compute the defining ideal of the 3rd Veronese surface. Paradoxically, this computation doesn't even "really" involve computing a Gröbner basis!

- If time allows, I'd like to also talk about **generic initial ideals** which have particularly nice properties. If I is a homogeneous ideal in a polynomial ring R, then after taking a generic change of coordinates and taking an initial ideal, the resulting monomial ideal J is Borel fixed, and has the same regularity and projective dimension as I. This amazing result allows you to prove, for instance, that if I has a linear resolution, then this must be the resolution of a Borel-fixed ideal, which is given by the Eliahou-Kervaire Resolution.

These notes are not meant to be exhaustive, or even precise. Gröbner bases can be technical, and my goal is to provide an overview and perhaps point you in the right direction should you some day be interested in problems where these techniques might be useful. As they notes progress, I'd be very happy to hear from you with any questions, comments or suggestions for new sections!

1.1 A Reading List

Perhaps the first place to learn about Gröbner bases is in the Undergraduate Textbook by Cox, Little, and O'Shea [1]. There are copious exercises, including lots of geometric examples. Eisenbud's book [2] has a very thorough treatment of Gröbner bases as well. Additionally, his treatment covers the more general setting of modules. My feeling is that this material is best seen first for ideals, but I recommend his book (and its exercises) as a great second course. Peeva's book Graded Syzygies

[3] contains many results in the spirit of these notes, including a very nice treatment of deformations and Gröbner basis theory, including applications to resolutions over non-regular rings. Finally, the book [4] by Miller and Sturmfels is another great source, especially if you are interested in what exactly can be said about the resolutions of monomial or toric ideals.

1.2 Where are we going? Three motivating questions.

When learning Gröbner bases it is unfortunately necessary to spend a lot of time actually doing messy computations, canceling terms, and pulling one's hair out looking for typos! Since my goal is to show how commutative algebraists can use Gröbner bases, I'll be skimming through these necessary details. I recommend using [1] for exercises on the basics, and once you have paid your penance, then Macualay2 is great for these sorts of things!

As a focus for these lectures, I wiIl focus on three questions:

Question 1.1. What are Gröbner bases and how should we think of them?

To me, this question is about motivation - why would one naturally be led to consider Gröbner bases and what features should they have? We'll begin by considering how a division algorithm might work for multivariate polynomials and see what subtleties arise. We will see that the natural object of an initial ideal arises.

Definition 1.2. Let $>$ be a monomial order. We define the leading term of a polynomial f to be the largest term $LT(f)$ in f. The set of all leading terms generates an ideal called the initial ideal of I.

$$\mathrm{in}_<(I) = \langle LT(f), \quad | \quad f \in I \rangle.$$

Question 1.3. What properties does $\mathrm{in}_< I$ share with I? Does the term order affect these properties?

Our main point, that will follow immediately from the division algorithm, is that Hilbert Series of I is equal to that of any initial ideal. While the proof of this is immediate, there are important ramifications, not the least of which is the non-obvious fact that any Hilbert Series is the Hilbert Series of a monomial ideal.

We will then begin to study the relationship between I and $\mathrm{in}_< I$ and see that in a precise sense, $\mathrm{in}_< I$ is a limit of a family of ideals, and as a degeneration there are semicontinuity theorems that once can prove. These theorems will concern the projective dimension, betti numbers, and regularity. To spoil the surprise, these invariants can increase as we pass to an initial ideal (but in a predictable way). In fact here's just some of the great properties we know:

- If I is any homogeneous ideal, then there is a monomial ideal with the same Hilbert Function as I.
- Let I be a homogeneous ideal and set $J = \mathrm{in}_< I$. Then if S/J (is Cohen-Macaulay, is a Complete Intersection, is of regularity $\leq d$) then so is S/I.

Given this semi-continuity, can we ask for more?

Question 1.4. Is there an initial ideal $\mathrm{in}_< I$ with the best behavior?

This vague question can lead in many possible directions, (for instance, the reader might investigage such things as Universal Gröbner Bases or more generally, robust ideals) but for these notes we'll focus on with generic initial ideals and a theorem of Bayer and Stillman. To spoil the whole show, we will prove that if I is an ideal, then if we take a generic change of coordinates and take the initial ideal with respect to a revlex term order, then we obtain a monomial ideal $\operatorname{gin} I$ with the same regularity and projective dimension as I.

2 What is a Gröbner Basis?

Throughout these notes k will be a field and $R = k[x_1, \ldots, x_n]$ will be a polynomial ring. We'll frequently change the names of the variables to suit our needs. We'll also frequently be using Macaulay2 for computations. You might want to begin by looking at Exercise 5. Our main motivation is the following:

Question 2.1. (The Ideal Membership Question) Given an ideal I and a polynomial f, can we determine if f is in I?

It is instructive to see what this question looks like over the simplest possible ring, $\mathbb{Z}$. How can we tell whether $17787 \in (43)$? The division algorithm allows us to write:

$$17787 = 28 + 413 \cdot 43$$

Since there is a nonzero remainder, we conclude that $17787 \notin (43)$. The statement is that if $a, b \in \mathbb{Z}$ then we can write

$$a = r + qb, \quad 0 \le r < a.$$

The same holds mutatis mutandis if $a(x), b(x)$ are polynomials of one variable - we can write

$$a(x) = q(x)b(x) + r(x), \quad 0 \le \deg r(x) < \deg a(x), \quad \text{or } r(x) = 0.$$

In either case, what is important is that a **size** has decreased. Iteratively applying this method, we get the **Euclidean algorithm** which will compute a greatest common divisor, $\gcd(a, b)$.

2.1 Monomial Term Orders and the Initial Ideal

Once we move on to polynomials in two or more variables our setup will have to be modified slightly. The problem is that we need a way to *compare* monomials in order to say that a remainder is smaller. For instance, is $x > y$? Is $y^2 > xz$? Or is $ab > z^5$? As we'll see, there are term orders that allow for all of these possibilities.

Definition 2.2. Let $>$ be a total order on the set of monomials in R. We say that $>$ is a monomial order if

1. Whenever $m_1 > m_2$ and x is a variable, then $xm_1 > xm_2$ (The order respects multiplication)
2. If $m \neq 1$ is a monomial then $m > 1$.
3. (Implied by (1) and (2) by Dickson's Lemma) The total order $>$ is a well-ordering. I.e. every nonempty set of monomials has a least element.

Example 2.3. Let's see some examples of monomial orders. If $x^a = x_1^{a_1} \cdots x_n^{a_n}$ then we can compare x^a and x^b. Evidently, a monomial order is an order on the exponent vectors.

1. Lexicographic Order (Lex): Assume that we have ordered $x_1 > x_2 > \ldots x_n$. We compare two monomials by comparing where they would appear in alphabetical order. $x^a > x^b$ iff the first nonzero entry of $a - b$ is positive. Note that this means e.g. that $a^3 > b^5$. (I will frequently alternate between using the letters of the alphabet and $x_1, x_2, \ldots$ without apology.) Here are the degree two monomials in a, b, c:

$$a^2 > ab > ac > b^2 > bc > c^2.$$

   Note: The Lex order depends on the choice of ordering the variables. For instance, there is a Lex order on $a > b > c$ and also one on the $b > a > c$. These orders are not the same, but they are both Lex orders.

2. Degree Lexicographic Order (DegLex): First use degree to determine which is bigger and then break ties by use the Lex order. For instance, with $a > b > \cdots > z$:

$$b^5 > a^3 \quad \text{and} \quad az^3 > b^4 > c^3 d$$

3. Degree Reverse Lexicographic (GradedRevLex): After first refining by degree, the mnemonic here is "least in the back", meaning that $x^a > x^b$ iff the last nonzero entry of $a - b$ is **negative**. For instance:

$$a^2 > ab > b^2 > ac > bc > c^2.$$

Note that as someone pointed out during my lecture, we won't get a term order without first breaking ties with total degree.

4. Weight order: We can give a (partial) term order by putting weights on each of the variables and then defining the weight of a monomial as the sum of the weights of the variables in the support. For instance, if $R = k[a, b, c]$ and we assign weights of $100, 10, 1$ to a, b, c then this will approximate the lex term order, since the weight of a^3 will be 300 whereas the weight of c^{200} is only 200. Thus $a^3 > c^{200}$. Of course this will never completely characterize Lex, but it's a decent approximation.

We will later use that if we only need to compare a finite list of monomials, then for any term order we can always choose a weight w so that the weight order from w agrees with $<$ for those pairs. Typically this finite list is all such monomials that could arise in a Gröbner basis calculation.

Recall that we have already defined the initial ideal of I to be the ideal generated by the leading terms of each polynomial in I. Notice that it does not suffice to take the leading terms of a generating set.

Example 2.4. Let I be the ideal of the twisted cubic.

$$I = I_2 \begin{pmatrix} x & y & z \\ y & z & w \end{pmatrix} = (xz - y^2, xw - yz, yw - z^2) =: (f_1, f_2, f_3).$$

The f_1 are minimal generators. Notice that $f_4 = xw^2 - z^3 \in I$ by inspection.

- If $>$ is the Lex term order induced by $x > y > z > w$ then we see that the leading terms of f_1, f_2, f_3 generate the ideal $J = (xz, xw, yw)$. Coincidentally $LT(f_4) = xw^2 \in J$. It terms out that $\text{in}_> I = J$ in this case. Later we'll see that we know this as soon as we see that J has the same Hilbert function as I.

- However, if we use a Lex term order with $y > x > z > w$ then the leading terms of f_1, f_2, f_3 generate the ideal $J = (y^2, yz, yw)$ which is only of height 1, and thus since I has height 2, it cannot be in the full initial ideal. Indeed, $LT(f_4) = xw^2 \notin J$. It turns out that $\text{in}_< I = (y^2, yz, yw, xw^2)$. With respect to this term order, we note that

$$\text{in}_< I = (LT(f_1), LT(f_2), LT(f_3), LT(f_4)).$$

  We say that $\{f_1, f_2, f_3, f_4\}$ is a Gröbner basis for this ideal and this monomial order.

Definition 2.5. Let I be an ideal in R and let $<$ be a monomial term order. A set of polynomials $\{g_i\}$ is called a Gröbner basis for I if

- Each $g_i \in I$
- $(LT(g_i)) = \text{in}_< I$.

In other words, it's a set of polynomials in I whose the leading terms generate the full initial ideal.

Proposition 2.6. *Let $<$ be a monomial term order. Then every ideal I has a Gröbner basis.*

Proof. Let $J = \text{in}_< I$. Then J is generated by some monomials $m_i = LT(g_i)$ with $g_i \in I$. By definition the g_i form a Gröbner basis. □

The following theorem shows that we can take this monomial generating set to be finite. It's worth visiting a proof of this result, say in [1, Section 2.4]. It's not "hard" but it's not "obvious."!

Theorem 2.7. *(Dickson's Lemma) Every monomial ideal is finitely generated. The monomial generators are unique.*

Once we have this we can in fact prove the Hilbert Basis Theorem.

Theorem 2.8. *(Hilbert Basis Theorem) The polynomial ring $R = k[x_1, \ldots, x_n]$ is Noetherian, i.e. every ideal is finitely generated.*

Proof. Let I be an ideal in R. Then I has a finite Gröbner basis $\{g_i\}$ by Dickson's Lemma. By Exercise 9, this shows that I is finitely generated. □

2.2 The Division Algorithm

In this section we describe a division algorithm for multivariate polynomials.

Proposition 2.9. *Let $G = \{g_1, \dots, g_r\}$ be a set of polynomials and let $f \in R$ be a polynomial. We define the following* ***division of*** f ***by*** G.

- *While a term of f is divisible by a $LT(g_i)$:*
  - *Subtract off an appropriate multiple of the g_i, (to cancel that leading term) to replace f with*
$$f := f - sg_i.$$
- *Return $f = r + \sum s_i g_i$ where no term of r is divisible by any of the $LT(g_i)$.*

This algorithm will terminate since we are requiring that $<$ is well-ordered, and by thinking through which terms are canceling in this algorithm.

It's worth pointing out that I'm intentionally ignoring what is going on with the leading coefficients of polynomials when defining the leading terms of polynomials. For instance, one could say that the leading term of $2x+1$ is $2x$ and that the leading monomial is x. This is important when defining the division algorithm so that things cancel, but I have chosen to ignore these technicalities for a more readable text since my goal is to present the spirit of the subject.

In almost all cases of interest, G will be a Gröbner basis for I with respect to a monomial term order $>$. The reason is:

Theorem 2.10. *Let $<$ be a monomial term order on R and let G be a Gröbner basis. Say $(G) = I$. Then if $f \in R$, the division algorithm of f divided by G will result in a unique remainder r. In particular $f \in I \iff r = 0$.*

Proof. We have that
$$f = r + \sum s_i g_i$$
and $LT(r)$ is not divisible by any of the $LT(g_i)$. (This doesn't require Gröbner anything). Suppose that
$$r = \sum s_i g_i = r' + \sum s_i' g_i'$$
where $r \neq r'$. Then $r - r' \in I$ and thus $LT(r - r') \in LT(I)$. This means that at least one term of r or r' is in $LT(I)$, and thus that term will be divisible by one of the $LT(g_i)$ (This uses that G is a GB), which is a contradiction. □

Remark 2.11. While the remainder upon the division algorithm is unique, the "quotients" s_i need not be unique.

2.3 Some Applications of the Division Algorithm

We have used the ideal membership problem as our motivation for Gröbner bases and the division algorithm. However, our proof that Gröbner bases exist is non constructive at the moment. Luckily there are effective algorithms for computing Gröbner bases .

Example 2.12. Compute a Gröbner basis for the ideal $I = (x^2, xy + y^2)$. With the order Lex on $x > y$.

Solution: Let $g_1 = x^2, g_2 = xy - y^2$. At the moment, our candidate for $\text{in}_< I$ is (x^2, xy). If the actual initial ideal is larger, then it must have generators not divisible by x^2 or xy. Thus, we can try and find elements in our ideal I without these as leading terms. We can do this, for instance by forcing these things away:

$$y(x^2) - x(xy - y^2) = xy^2 \in I.$$

Now let's perform the division algorithm on xy^2 by $\{g_1, g_2\}$.

$$xy^2 = y^3 + y(xy - y^2)$$

and our remainder is y^3. If $\{g_1, g_2\}$ was a Gröbner basis then we should have gotten a remainder of 0. Let's then add $g_3 = y^3$ to our candidate for our Gröbner basis, $G = \{g_1, g_2, g_3\}$.

To recap:

- We took two polynomials g_1, g_2, took a linear combination to cancel their leading terms.
- We took the result and divided it by $\{g_1, g_2\}$.
- If the remainder is nonzero, call it g_{new} and add it to the list G.

Theorem 2.13. *(Buchberger's Algorithm) If $I = (g_1, \ldots, g_r)$ and $<$ is a monomial term order, then we say the S-pair between g_i and g_j is the polynomial*

$$S(g_i, g_j) = \frac{LT(g_j)}{GCD}(g_i) - \frac{LT(g_i)}{GCD}(g_j)$$

where $GCD = \gcd(LT(g_i), LT(g_j))$. Essentially this S-pair is exactly the minimal way to cancel the leading terms.

Algorithm:

- *Let G be the current set of g_i.*
- *While some $S(g_i, g_j)$ has nonzero remainder g_{new} when divided by G, add g_{new} to G.*

This algorithm terminates and returns a Gröbner basis G.

Note, that this means that as we add new elements to the set G, we still have to go back and check that the S-pairs "reduce to zero" with the new set. In the context of our previous example, we would have to "check (1,2), add 3, check (1,3), check(2,3)."

2.4 Initial Ideals Give k-bases

Theorem 2.14. *Let I be an ideal, and $<$ a monomial term order. Then the set of monomials NOT in $\mathrm{in}_< I$ is a k-basis for both R/I and $R/\mathrm{in}_< I$.*

Proof. The result is clear for $R/\mathrm{in}_< I$. We prove that they are a basis for R/I.

Independence Suppose that $\sum a_i \overline{m_i} = 0$ for monomials m_i and $a_i \in k$. Then this means that $\sum a_i m_i = f \in I$. But this means that $LT(f)$ must be one of the $a_i m_i$. (Note that this proof only requires that $LT(f)$ selects a term from f).

Spanning It will be sufficient to prove that each $f \in R$ is equivalent modulo I to something in the span of monomials not in $\mathrm{in}_< I$. To that end, divide f by a Gröbner basis of I. Then the remainder will by definition be a sum of monomial not in the initial ideal. □

Example 2.15. Let $I = (x^2 - y) \subset R = k[x, y]$. Then I has two initial ideals (x^2) and (y). Hence the following two sets are bases for R/I:

$$\{1, x, x^2, x^3, x^4, \ldots\} \text{ is a basis for } R/I \text{ and also for } R/(y).$$

$$\{1, x, y, xy, xy^2, xy^3, \ldots\} \text{ is a basis for } R/I \text{ and also for } R/(x^2).$$

It's worth noting that the degrees of these basis monomials are different in these cases. We'll see that this doesn't happen in the case that I is homogeneous.

Example 2.16. Let $I = (x^2, xy+y^2)$. Then I has two initial ideals, namely (x^2, y^2) and (x^2, xy, y^3). If we write down the corresponding monomial bases for R/I we get

$$\begin{array}{c|c} 1 & 1 \\ x, y & x, y \\ xy & y^2 \end{array}$$

Example 2.17. Let I be the ideal of the twisted cubic:

$$I = I_2 \begin{pmatrix} x & y & z \\ y & z & w \end{pmatrix} = (xz - y^2, xw - yz, yw - z^2).$$

With the ordering $y > x > z > w$

$$\mathrm{in}_{revlex} I = (y^2, yz, z^2), \quad \mathrm{in}_{lex} I = (y^2, yz, yw, xw^2)$$

And we have the following monomial bases for R/I:

$$\begin{array}{c|c} 1 & 1 \\ x, y, z, w & x, y, z, w \\ x^2, xy, xz, xw, yw, zw, w^2 & x^2, xy, xz, xw, z^2, zw, w^2 \\ \text{(10 monomials)} & \text{(10 monomials)} \\ \cdots & \cdots \end{array}$$

In fact, I has 8 different initial ideals! Although there are infinitely many different monomial term orders, it is true that there will only be finitely many initial ideals. If you want to find them all for a given ideal, you can type

```
loadPackage "gfanInterface"; gfan I
```

into Macaulay2.

In the previous example, I was homogeneous, which allowed us to sort the basis elements according to their degrees. This process encodes the **Hilbert Function** of R/I which will be our next object of study.

2.5 Initial Ideals Have the Same Hilbert Function

If $R = k[x_1, \ldots, x_n]$ and M is a finitely generated graded R-module then the **Hilbert Function** of M is a function $HF_M : \mathbb{Z} \to \mathbb{Z}$ given by

$$HF_M(d) = \dim_k(M_d)$$

the dimension of the k-vector space M_d. It is known that $HF_M(d)$ is eventually polynomial in d, and the degree of this polynomial is one less than the dimension of the module M. A bit more is true - the generating function for HF_M, called the **Hilbert Series** is actually a rational function

$$HS(M) := \sum HF_M(d)\ t^d = \frac{f(t)}{(1-t)^D}, \quad \dim M = D.$$

where $f(t)$ is a polynomial in t, t^{-1} and $f(1) \neq 0$.

Theorem 2.18. *Let I be a homogeneous ideal in R, and let J be any* ***initial ideal*** *of I. Then*

$$HS(R/I) = HS(R/J).$$

In particular, the Hilbert function $HF(I)$ is the Hilbert function of a monomial ideal.

Proof. This follows immediately from the following two facts:

- If I is homogeneous, then the subspace of degree i homogeneous polynomials in R/I is spanned by monomials.
- There is a basis of monomials that is simultaneously a basis for R/I and R/J. (Theorem 2.14).

□

Corollary 2.19. *If I is a homogeneous ideal in R and J is any initial ideal of I, then* $\dim S/J = \dim S/I$.

Given any homogeneous ideal I, there is a monomial ideal with the same Hilbert function. This shows that the Hilbert functions of monomial ideals are no more special that those of arbitrary graded ideals. However, the same cannot be said for instance, about the degrees and number of generators for an ideal. We have already seen that before:

$$HS(R/(x^2, y^2)) = HS(R/(x^2, xy, y^3)).$$

These two ideals have different numbers of minimal generators, despite having the same Hilbert function.

It is even true that for any N there exists an ideal with 3 generators such that the initial ideal requires at least N minimal generators.

We close this section with one statement and some questions.

Proposition 2.20. *Suppose that I is a homogeneous ideal that is minimally generated by r generators. Then any initial ideal of I requires at least r generators.*

Proof. Let $<$ be a monomial term order. Let $J = \text{in}_< I$ be the initial ideal. Suppose that J requires s minimal generators. Then there is a Gröbner basis G with s polynomials. By Exercise 9, G is a generating set for I. □

> If μ denotes the number of minimal generators then
>
> $$\mu(I) \leq \mu(\text{in}_< I).$$

Question 2.21.

1. Given a Hilbert Series ϕ, is there an ideal with Hilbert Series ϕ with the largest number of generators?

2. Since

   $$\text{number of minimal generators of } I \text{ of degree } j = \beta_{1j}(R/I),$$

   what can we say about the relationship between the Betti numbers of R/I and R/J when J is an initial ideal of I?

3. Are there examples of ideals I such that all initial ideals have the same number of generators? The same betti numbers? (Vague open-ended question)

The answer to Question (1) is tangential to the focus of these lectures, but we'll state it here for completeness. Given Proposition 2.20, the ideal with the largest number of minimal generators will be monomial (if it exists). Given a Hilbert function, you can think of this as telling you how many monomials are in your ideal in each degree. A wild dream might be that given such a function, if you just took the lexicographically first such monomials in each degree, then the ideal they generate might have your Hilbert function. This wild dream turns out to be correct and was proven in 1927 by Macaulay. This ideal is called the **Lex Ideal** of that Hilbert Series.

Example 2.22. Let $I = (x^3, y^3)$ in $k[x, y, z]$. Then the Lex Ideal of I is

$$L = (x^3, x^2y, x^2z^2, xy^3z, xy^4, xy^2z^3, xyz^5, xz^7, y^9)$$

The betti tables of R/I and R/L are as follows:

R/I			
	1	–	–
	–	–	–
	–	2	–
	–	–	–
	–	–	1

R/L				
	1	–	–	–
	–	–	–	–
	–	2	1	–
	–	1	2	1
	–	2	3	1
	–	1	2	1
	–	1	2	1
	–	1	2	1
	–	1	1	–

Theorem 2.23. *Let ϕ be the Hilbert series of some quotient of R be a homogeneous ideal. Let L be the Lex ideal described above. Then if I is* any *ideal with $\phi = HS(R/I)$ then*

1. *$\phi = HS(R/L)$ (i.e. L actually has the Hilbert function it's supposed to have) (Macaulay)*
2. *For each j,*

   number of minimal generators of I of degree $j \leq$ number of minimal generators of L of degree j

   (i.e. the Lex ideal has the largest number of generators in a strong sense) (Macaulay)
3. *For each i, j,*

$$\beta_{ij}(R/I) \leq \beta_{ij}(R/L)$$

   (this is due to Bigatti, Hulett, Pardue).

Note that it is not obvious that taking the Lex-first monomials results in an ideal with the right Hilbert function. Indeed, is false if ϕ is not an actual Hilbert function. For instance, suppose that $R = k[a, b, c]$ and $\phi = 1 + 3t + 4t^2 + 4t^3 + 6t^4 + \dots$. Monomials are drawn below with the horizontal separation indicating whether they are in L or R/L:

L	R/L	$\dim_k(R/L)$
	1	1
	a, b, c	3
a^2, ab	ac, b^2, bc, c^2	4
$a^3, a^2b, a^2c, ab^2, abc, ac^2$	b^3, b^2c, bc^2, c^3	4
$a^4, a^3b, a^3c, a^2b^2, a^2bc, a^2c^2, ab^3, ab^2c, abc^2$	$ac^3, b^4, b^3c, b^2c^2, bc^3, c^4$	5

Our candidate for L has a problem, namely, since $ac^2 \in L$ then so should ac^3. If you think about this statement, part of what Macaulay's Theorem says is that if our ϕ is the Hilbert Series of $k[a, b, c]/I$ then if it is $1 + 3t + 4t^2 + 4t^3 + pt^4 + \cdots$ then $p \leq 5$. The moral of this boxed statement is that Macaulay's theorem is (in part) about the growth of Hilbert functions!

3 Gröbner bases and Free Resolutions

We follow Section 22 of [3] closely for portions of this section.

Definition 3.1. A minimal free resolution of an finitely generated graded R-module M is an exact sequence of graded maps of free R-modules,

$$0 \longrightarrow F_r \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0.$$

where the ranks of the F_i are chosen to be as small as possible. Equivalently, the resolution is minimal if the maps ∂ in the resolution satisfy that $\partial(F_i) \subset (x_1, \ldots, x_n)F_{i-1}$, i.e. that the matrices have no entries that are units. In this case, we can write F_i as a direct sum of $R(-j)^{\beta_{ij}}$ for some nonnegative integers β_{ij} called the Betti numbers of M.

So far we have seen that if I is an ideal and J is an initial ideal of I then the Hilbert Series (and thus the dimension and degree) of R/I is equal to that of R/J. The set of **graded betti numbers** is a strict refinement of the Hilbert Series, however, and our next step is to study what role Gröbner bases play here. We return to our favorite example:

Example 3.2. Let $I = (x^2, xy + y^2)$ with the monomial term order induced by $x > y$. Then $J = (x^2, xy, y^3)$ is the initial ideal of I. Let's compute the minimal free resolutions of R/I and R/J.

$$0 \longrightarrow R(-4) \xrightarrow{\begin{bmatrix} xy+y^2 \\ -x^2 \end{bmatrix}} R(-2)^2 \xrightarrow{\begin{bmatrix} x^2 & xy+y^2 \end{bmatrix}} R \longrightarrow R/I$$

$$0 \longrightarrow \begin{matrix} R(-4) \\ \bigoplus \\ R(-3) \end{matrix} \xrightarrow{\begin{bmatrix} 0 & y \\ y^2 & -x \\ -x & 0 \end{bmatrix}} \begin{matrix} R(-2)^2 \\ \bigoplus \\ R(-3) \end{matrix} \xrightarrow{\begin{bmatrix} x^2 & xy & y^3 \end{bmatrix}} R \longrightarrow R/J$$

The graded betti numbers are thus:

$$R/I: \quad \beta_{12} = 2, \quad \beta_{24} = 1$$

$$R/J: \quad \beta_{12} = 2, \beta_{13} = 1 \quad \beta_{23} = 1, \quad \beta_{24} = 1$$

We present this data in a betti table:

R/I			
	1	–	–
	–	2	–
	–	–	1

R/J			
	1	–	–
	–	2	1
	–	1	1

What's going on with $R(-j)$? The shifts in these resolutions (things like $R(-3)$) are unfortunately rather confusing, but essentially for the right numerical information to work out. What is going on is roughly the following: Consider the map that is multiplication by x^5.

$$\phi : R \xrightarrow{\cdot x^5} R.$$

This map is NOT a graded map under the naive grading. Indeed, the degree d polynomial x^d goes to the the degree $d+5$ polynomial x^{d+5}. To solve this, we will have to adjust the grading on either the source or the target. We'll choose the convention of adjusting the source. We'll look at

$$\phi(x^d) = x^{d+5}$$

and agree that the degree of x^d should be $d+5$. In other words, our grading on our source should be 5 degrees lower than expected. Hence we define $R(-5)$ to be the graded ring whose generator lies in degree 5. In terms of grading,

$$R(-5)_i = R_{i-5}$$

We notice two things with these resolutions:

1. The first is "contained" inside of the other, at least numerically.
2. The extra factors occur in "canceling homological degrees".

Let's see what we mean by the second statement. We compute the Hilbert series as in Exercise 22 and see that it is equal to

$$HS(R/I) = \frac{1-2t^2+t^4}{(1-t)^2}, \quad HS(R/J) = \frac{1-(2t^2+t^3)+(t^3+t^4)}{(1-t)^2}$$

Thus we can see that these cancellations are precisely what ensures that the Hilbert function is the same.

Now our term order $x > y$ is actually a weight order with the weights of x and y being 1 and 0. We can **homogenize** the ideal I with respect to this new term order by adding a new variable u and using u to jack up the weights of smaller terms. For instance, the homogenization of $xy + y^2$ would be $xy + y^2u$. If we let $\tilde{I}$ be the ideal generated by the homogenizations of all polynomials in I, then in this case

$$\tilde{I} = (x^2, xy + y^2u, y^3).$$

If we resolve $R[u]/\tilde{I}$ then we obtain

$$0 \longrightarrow \begin{matrix} R[u](-4) \\ \bigoplus \\ R[u](-3) \end{matrix} \xrightarrow{\begin{bmatrix} 0 & y \\ y^2 & -x+yu \\ -x-uy & -u^2 \end{bmatrix}} \begin{matrix} R[u](-2)^2 \\ \bigoplus \\ R[u](-3) \end{matrix} \xrightarrow{\begin{bmatrix} x^2 & xy+y^2u & y^3 \end{bmatrix}} R[u] \longrightarrow R[u]/\tilde{I}.$$

Notice

1. This is a graded resolution with respect to the weight order and also ignoring the u.
2. If we set $u = 1$ then we obtain a (non-minimal) resolution of R/I
3. If we set $u = 0$ then we obtain a (minimal) of R/J.

We will set up the generalities of this now.

3.1 Homogenization of Ideals

We begin with a lemma about nonzero divisors and that they do not affect free resolutions.

Lemma 3.3. *Suppose that $F_\bullet \to M$ is a free resolution of M and $x \in R$ is M-regular. Then $F_\bullet \otimes R/(x)$ is a free resolution of M/xM as an $R/(x)$-module.*

Proof. Note that
$$H(F_\bullet \otimes R/(x)) = \mathrm{Tor}(M, R/(x)) \cong \mathrm{Tor}(R/(x), M).$$
But now a resolution of $R/(x)$ is simply $R \xrightarrow{x} R$ and so
$$\mathrm{Tor}(R/(x), M) = H(\, M \xrightarrow{x} M\} \, = M/xM$$
since x is M-regular. ☐

Corollary 3.4. *Let $\tilde{I}$ be an ideal in $R[u]$. Suppose that u and $u-1$ are regular elements on $R[u]/\tilde{I}$ and*
$$R[u]/(u, \tilde{I}) \cong R/I_0, \qquad R[u]/(u-1, \tilde{I}) \cong R/I_1.$$
Then if $\tilde{F}$ is a minimal free resolution of $R[u]/\tilde{I}$ then

1. *$\tilde{F} \otimes R/(u)$ is a **minimal** free resolution of R/I_0*
2. *$\tilde{F} \otimes R/(u-1)$ is a free resolution of R/I_1.*

Proof. Exactness follows from the Lemma. For the statement on minimality, notice that setting $u = 0$ will not result in any of the entries in the matrices becoming units. However, this may occur if we set $u = 1$. ☐

As motivation for homogenizations, we talk about projective closure. We illustrate this with an example.

Example 3.5. Suppose that $I = (y - x^2, z - x^3)$ is the "affine twisted cubic", i.e. the ideal of the locus X of all points of the form $\{(t, t^2, t^3)\}$ in $\mathbb{A}^3$. Our goal is to compute the defining equations in $k[x, y, z, u]$ for the projective closure of X in $\mathbb{P}^3$. The ideal defining the projective closure should be
$$\tilde{I} = (\tilde{f} \mid f \in I)$$

where $\tilde{f}$ is the polynomial obtained by adding appropriate factors of the new variable u to all terms in f not of maximal degree. If you like equations, it goes something like this:

$$\tilde{f} = u^{\deg f} f\left(\frac{x_i}{u}\right).$$

For instance, if we homogenize, $y - x^2$ we obtain $yu - x^2$. Now it will not be sufficient to homogenize any generating set for I, but if we compute a Gröbner basis for any term order that refines the degree, (say, DegRevLex) then if we homogenize the elements of that Gröbner basis, those will generate the ideal for the projective closure of X. In our case,

```
S = QQ[x,y,z, MonomialOrder=> GRevLex];
I = ideal(x^2 -y, x^3 - z);
gens gb I
```

Output: `|y2-xz, xy-z, x2-y|`

If we homogenize these generators we obtain $(y^2 - xz, xy - uz, x^2 - yu)$, which is the ideal for the projective closure of X.

Proposition 3.6. *If I is an ideal, and $<$ is a monomial term order that refines degree, and G is a Gröbner basis for I with respect to $<$ then the homogenization of I is given by*

$$\tilde{I} = (\tilde{g} \mid g \in G).$$

The situation for homogenization with respect to a weight vector is analogous - the only difference is that rather than adding factors of u to jack up the degree, we want to increase the weight of each term. Indeed, suppose that w is weight vector, which assigns to each monomial a weight. Then set $w(f)$ to be the largest weight of any term of f. Then the homogenization of a polynomial f is given by

$$\tilde{f} = u^{w(f)} f\left(\frac{x_i}{u^{w(x_i)}}\right).$$

Example 3.7. If $w = (6, 5, 4)$ and $f = x^3 - xy + z^5$ then $w(f) = 20$ and

$$\tilde{f} = x^3u^2 - xyu^9 + z^5.$$

Definition 3.8. Let I be an ideal and let $<$ be a monomial term order, and let G be a Gröbner basis with respect to $>$. If w is a weight order that is equivalent to $>$ on all the terms appearing in G, then let $\tilde{I}$ be the homogenization of I described above. Note that if $f \in I$ then $\tilde{f}$ will be the unique term of f without the parameter u.

Note that $\tilde{I}$ will be weighted-homogeneous. We will think of u as being a parameter, denote by I_α the image of $\tilde{I}$ in $R[u]/(u - \alpha)$. Evidently $I_0 = \text{in}_< I$ and $I_1 = I$. The following is proven in Eisenbud [2, Theorem 15.17].

Theorem 3.9. *$R[u]/\tilde{I}$ is free as a $k[u]$-modules and thus all elements $u - \alpha$ are regular elements.*

Corollary 3.10. *Suppose that I is a homogeneous ideal and $<$ is a monomial term order. Then*

1. $\beta_{ij}(R/I) \leq \beta_{ij}(R/\operatorname{in}_< I)$ *for all* i, j*;*
2. $\operatorname{pdim} R/I \leq \operatorname{pdim} R/(\operatorname{in}_< I)$*;*
3. $\operatorname{reg} R/I \leq \operatorname{reg} R/(\operatorname{in}_< I)$*;*

Proof. Recall that

$$\operatorname{pdim}(M) = \max\{i \mid \beta_{ij}(M) \neq 0 \text{ for some } j\}$$

and

$$\operatorname{reg}(M) = \max\{j - i \mid \beta_{ij}(M) \neq 0\}$$

so the result follows once we prove (1). To prove this, let $F_\bullet$ be a minimal free resolution of $\tilde{I}$ that is homogeneous with respect to the weight order w. It will naturally be homogeneous with respect to the usual $\mathbb{Z}$-grading as well. By Theorem 3.9, and Lemma 3.3, $F_\bullet \otimes R[u]/(u - \alpha)$ will be a resolution for all α. It will be minimal precisely when the maps all have entries in the maximal ideal. This definitely happens when $\alpha = 0$. Thus:

$$\beta_{ij}(R/\operatorname{in}_< I) = \beta_{ij}(R/I_0) = \beta_{ij}(F_\bullet \otimes R[u]/u)_i) = \beta_{ij}(F_\bullet).$$

When $\alpha = 1$ we $F_\bullet \otimes R[u]/(u-1)$ will still be exact, but it may be a nonminimal resolution. But since any resolution can be minimalized, this shows that

$$\beta_{ij}(R/I) = \beta_{ij}(R/I_1) \leq \beta_{ij}(F_\bullet \otimes R[u]/u)_i) = \beta_{ij}(F_\bullet).$$

The desired inequality in (1) holds. □

3.2 From Consecutive Cancellations to the Generic Initial Ideal

At this point we have seen that there is a resolution of $R[u]/\tilde{I}$ that specializes to resolutions of both R/I and $R/\operatorname{in}_< I$. Since the resolution of the initial ideal can be bigger, it's worth asking how much bigger.

Proposition 3.11. *(Theorem 20.2 in Eisenbud) Suppose that M is a f.g. graded module and that $F_\bullet$ is a minimal free resolution of M. Then any free resolution of M is isomorphic to the direct sum of $F_\bullet$ and a trivial complex, i.e. the direct sum of complexes of the form*

$$\cdots \longrightarrow 0 \longrightarrow R \xrightarrow{1} R \longrightarrow 0 \longrightarrow \cdots .$$

Corollary 3.12. *The Betti numbers of $R/\operatorname{in}_<(I)$ differ from those of R/I by those of a trivial complex, in other words, they differ by a* ***consecutive cancellation.***

It's probably easiest to illustrate what a consecutive cancellation is in terms of Betti tables. The point is that if $\beta_{ij}(R/\operatorname{in} I) > \beta_{ij}(R/I)$ then so must also $\beta_{i+1,j}$ or $\beta_{i-1,j}$ be one bigger as well. For instance, if I is the ideal of the twisted cubic, then we have seen an initial ideal where the betti tables are:

R/I			
	1	–	–
	–	2	–
	–	–	1

R/J			
	1	–	–
	–	2	$\textcircled{1}$
	–	$\textcircled{1}$	1

Example 3.13. Let $I = (ab, bc, cd, de, ae) \subset k[a, b, c, d, e]$. Now obviously, $I = \text{in}_< I$ for all term orders $<$. But notice that if we do a generic change of coordinates, say,

$$a \to 5a - 6b - \frac{1}{3}c + 17d - e$$

$$b \to -a + b - c + 7d - 6e$$

$$\dots$$

then we would obtain an $g(I)$ ideal that isomorphic to I. We think of $g \in GL_5 k$ as acting (via automorphisms) on R via change of coordinates. However, $g(I)$ will how be far from monomial. It's initial ideal with respect to RevLex will be the ideal $J = (a^2, ab, b^2, ac, bc, c^3)$. We call this ideal $J = \text{gin}(I)$ the **generic initial ideal** of I. Here are some betti tables:

R/I				
0	1	–	–	–
1	–	5	5	–
2	–	–	–	1

$R/\text{gin}(I)$				
0	1	–	–	–
1	–	5	(6)	((2))
2	–	(1)	((2))	1

Here the number of circles indicates the number of successive cancellations that occur in the degeneration from I to $\text{gin}(I)$. (Mostly it's my amusement with the circling code I stole from StackExchange)

Notice

1. The number of rows in each betti table is the same. This number is called the Castelnuovo-Mumford Regularity of R/I. Note that it is (one less than) the degree of the largest monomial generator of $\text{gin}(I)$
2. The projective dimension of R/I is equal to that of $R/\text{gin}(I)$.

The following theorem follows from Exercise 3:

Theorem 3.14. *(Bayer-Stillman) Let $R = k[x_1, \dots, x_n]$. Suppose that $>$ is RevLex. Then elements $x_n, x_{n-1}, \dots, x_i$ form a regular sequence on R/I if and only if they do on $R/\text{in}_< I$.*

Theorem 3.15. *(Galligo,Bayer-Stillman) Inside of $GL_n(k)$ there is a Zariski-open set U, and a monomial ideal J such that for all $g \in U$, $\text{in}_{revlex}\, g(I) = J$. We denote J by $\text{gin}(I)$. The generic initial ideal is fixed by the Borel group B of upper triangular invertible matrices, meaning that if $g \in B$ then $g(\text{gin}(I)) = \text{gin}(I)$.*

For a thorough treatment of generic initial ideals, [2] does the general case, and [4] gives a presentation in terms of Parametric Gröbner bases. The basic idea is that one could write an element of $GL_n(k)$ as a tuple of n^2 indeterminates g_{ij} and then $g(I)$ would be an ideal in $R[g_{ij}]$. One could apply Buchberger's algorithm, tracing through, though at each step keeping track of things like "oh if this coefficient is zero, then the leading term will actually be this term, so we would multiply by this...". If at each step we assume the coefficients are nonzero, this is a condition on the tuple g_{ij} and this is the Zariski-open set.

Theorem 3.16. *(Bayer-Stillman) Let I be a homogeneous ideal. Then*

1. *The Castelnuovo-Mumford Regularity of R/I is equal to that of $R/\operatorname{gin}(I)$. If the characteristic of k is zero, then this number is (one less than) the biggest degree of a generator of $\operatorname{gin}(I)$.*

2. *The projective dimension of R/I is equal to that of $R/\operatorname{gin}(I)$.*

3. $$\operatorname{depth}(R/I) = \operatorname{depth}(R/\operatorname{gin}(I)) = \max\{t \mid x_n, x_{n-1}, \ldots, x_{n-t+1} \notin \operatorname{gin}(I)\}.$$

Proof. The paper of Bayer and Stillman [8] is short, and hopefully now very accessible. For the result on the regularity, we refer the reader there.

We will prove the statement on depth, which by the Auslander-Buchsbaum theorem is equivalent to the statement on projective dimension. Recall that we have already showed that

$$\operatorname{pdim}(R/I) \leq \operatorname{pdim}(R/\operatorname{gin}(I)).$$

By Auslander-Buchsbaum, this implies that

$$\operatorname{depth}(R/I) \geq \operatorname{depth}(R/\operatorname{gin}(I)).$$

Suppose that the depth of R/I is s. If $s = 0$ we are done. Suppose $s \geq 1$. Then by prime avoidance, a sequence of s generic linear forms will be a regular sequence on R/I. Since after a generic change of coordinates, these linear forms will be variables, we may suppose that $x_r, \ldots, x_{r-s+1}$ is a regular sequence. But by Theorem 3.14, we have that this will be a regular sequence on $R/\operatorname{gin}(I)$ (remember that we defined $\operatorname{gin}(I)$ using RevLex. Thus $\operatorname{depth}(R/\operatorname{gin}(I)) \geq s$. Thus

$$s = \operatorname{depth}(R/I) \geq \operatorname{depth}(R/\operatorname{gin}(I)) \geq s$$

and the equality holds. □

4 Exercises

4.1 Exercises for Day 1

Exercise 1. Prove that if J is an initial ideal of I and J is reduced then I is reduced. Is the converse true?

Exercise 2. Let $I = (x^2, y^2, ax + by) \subset k[a, b, x, y]$.

1. Compute a Gröbner basis for I with respect to the Lex term order on $a > b > x > y$.
2. What is the initial ideal of I?
3. Use Macaulay2 to compute the minimal free resolution of R/I

   ```
   R = QQ[a,b,x,y]; I = ideal"x2,y2,ax+by"; betti res I
   ```
4. What is the resolution of the initial ideal?
5. Do you see any relationship between the betti tables?

Exercise 3. Let $R = k[x_1, \ldots, x_n]$ and suppose that $>$ is any term order.

1. Show that if a variable x is a regular element on $R/\operatorname{in}_< I$ then x is a regular element on I. Find an example where the converse fails: find an ideal I such that x is regular on R/I but not regular on $R/\operatorname{in}_< I$. (Hint: Find a principal ideal)
2. Now let $<$ be RevLex. Prove that the last variable x_n is a regular element on R/I if and only if it is a regular element on $R/\operatorname{in}_< I$.

This exercise can be extended with just a few more steps to the following result, first proved by Bayer and Stillman (*Inventiones* 1987). You're basically there!

Theorem 4.1. *(Bayer-Stillman)Let $R = k[x_1, \ldots, x_n]$. Suppose that $>$ is revlex. Then elements $x_n, x_{n-1}, \ldots, x_i$ form a regular sequence on R/I if and only if they do on $R/\operatorname{in}_< I$.*

Exercise 4. By finding an appropriate term order, can you prove with no computation that the ideal

$$I = (a^2 + b^2 + c^2 + d^2, d^4 + abcd + c^4, a^2 - 2ac + c^2)$$

has height 3? (And is thus a complete intersection?)

Exercise 5. Draw the projective plane $\mathbb{RP}^2$ by drawing a circle and identify opposite sides antipodally. By introducing points on the circle and interior, draw a **triangulation** of $\mathbb{RP}^2$. This means that you will split the surface into triangles such that the triangles all have three distinct vertices and that those vertices determine (at most) one triangle. (Hint: Label your vertices a, b, c, d, e, f).

- Now that you have your triangles, write down the set of squarefree degree three monomials that do *not* correspond to any triangle. Let I be the ideal that these generate. This is called the Stanley-Reisner Ideal of this triangulation.

- Input this ideal into Macaulay2 using the commands:

```
R = QQ[a,b,c,d,e,f]
I = ideal"abc,def,???"
codim I, betti res I
```

- Now try it over a different field:

```
R = ZZ/2[a,b,c,d,e,f]
J = ideal"abc,def,???"
codim J, betti res J
```

- Work out the Hilbert Series of R/I from the betti table. You can check this using `HilbertSeries(I, Reduce=>true)`. You'll find that

$$HS = \frac{1 + 3T + 6T^2}{(1 - T)^3}$$

The numerator is called the **h-polynomial**, for "Hilbert". Note that

$$1 + 3T + 6T^2 = 6(T - 1)^2 + 15(T - 1) + 10.$$

The coefficients on the right $(6, 15, 10)$ are called the **f-vector**, for "face." Notice that your picture has 6 vertices, 15 edges and 10 triangles.

Note: When inputting an ideal you can either type `x^2*y^3+3*x*y` with the carats and stars. Or else you can input `ideal"x2y3+3xy"`.

Exercise 6.

1. Suppose that $I \subset J$ are homogeneous ideals so that $\dim R/I = 0$. In other words, R/I and R/J have finite length. Prove that if they have the same length, then they are in fact equal.
2. More generally, prove that if $I \subset J$, and I is radical and equidimensional then if the **degree** of R/I equals the degree of R/J then $I = J$. (By degree, we mean the multiplicity).

Hints: Suppose that $J = Q_1 \cap \cdots \cap Q_r \cap K$ is a primary decomposition where the Q_i are primary to minimal primes and K is everything else. Suppose that $I = P_1 \cap \cdots \cap P_s$.

- Show that each $\sqrt{Q_i}$ must be one of the $P_i's$.
- Without loss of generality, say $P_1 = \sqrt{Q_1}$. Then localizing at P_1 we have

$$(P_1)R_{P_1} \subset (Q_1)R_{P_1}.$$

Prove that this means that $Q_1 = P_1$.

- Use the fact that $\text{degree}(R/I) = \sum \text{degree}(R/P_i)$, and $\text{degree}(R/J) = \sum \text{degree}(R/Q_i)$, to prove that if $s = r$ and therefore $J = P_1 \cap \cdots \cap P_r \cap K$.
- Conclude $K = (1)$ and thus $I = J$.

Exercise 7. Let $I = (xy, yz, xz)$. Compute $HS(M)$ with $M = S/I$ and $M = I$. Hint: What is the relationship between $HS(S/I)$ and $HS(I)$? It might help to write a basis for the degree d piece of M in low degree.

4.2 Exercises for Day 2

Exercise 8. Let $R = k[x_1, \dots, x_n]$ and suppose that $>$ is any term order. Prove the Theorem of Bayer-Stillman

Let $R = k[x_1, \dots, x_n]$. Suppose that $>$ is revlex. Then elements $x_n, x_{n-1}, \dots, x_i$ form a regular sequence on R/I if and only if they do on $R/\operatorname{in}_< I$.

Exercise 9. Prove that if G is a Gröbner basis for I then it is a generating set for I.

Exercise 10. Let I be the ideal of the twisted cubic

$$I = (xz - y^2, xw - yz, yw - z^2)$$

assign the weights $(0, 10, 5, 1)$ to (x, y, z, w).

1. Compute the homogenization $\tilde{I}$ of I with respect to these weights. (Hint: it will have four generators)

2. Compute the minimal free resolution of $R[u]/\tilde{I}$.

   If using Macaulay2, it will expect things to be homogeneous. You can check if it thinks something is homogeneous by typing: `isHomogeneous f`. You can add weights to variables e.g. `R = QQ[x,y,z,w,Degrees=>{0,10,5,1}]`. You could also add a monomial order like:

   ```
   R = QQ[x,y,z,w,Degrees=>{0,10,5,1},MonomialOrder=>{Weights=> {0,10,5,1}}]
   ```

Exercise 11. Consider the image of the map $\mathbb{P}^2 \to \mathbb{P}^9$ given by

$$[a : b : c] \mapsto [a^3 : a^2b : \cdots : c^3].$$

It is defined by the ideal I given by $I = \ker(\phi : k[x_1, \dots, x_{10}] \to k[a, b, c])$ with

$$\phi(x_1) = a^3, \ \dots \ , \phi(x_{10}) = c^3.$$

1. Note that $V = k[x_1, \dots, x_{10}]/I \cong k[a^3, \dots, c^3]$ the so-called 3rd Veronese subring of $k[a, b, c]$. Use this to compute the dimension of V and thus the codimension of I.

2. It is true that the degree of V is equal to 9. In general, geometrically, if we embed $\mathbb{P}^n$ using the divisor that is d times a hyperplane section, then the degree of the image will be n^d. It's possible to prove this algebraically, say, by computing the Hilbert Series, which was done in [5] and [6].

3. What are the generators of I? Who knows! Consider the following matrix:

$$M = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ x_2 & x_4 & x_5 & x_7 & x_8 & x_9 \\ x_3 & x_5 & x_6 & x_8 & x_9 & x_{10} \end{pmatrix}$$

   Figure out why the 2×2 minors of this matrix are all in I. Hint: Consider the following picture which suggests a "factorization" of the cubic polynomials."

	a^2	ab	ac	b^2	bc	c^2
a	1	2	3	4	5	6
b	2	4	5	7	8	9
c	3	5	6	8	9	10

4. Now using the Lex term order with $x_1 > \ldots > x_{10}$ compute the initial terms of each of these minors. They generate an ideal K. Luckily (!!!) this ideal is nice enough that we can compute its primary decomposition using Macaulay2 (this is an easy combinatorial algorithm) and see that it has 9 components, all of height 7. Can you find a combinatorial proof of this?

5. Using Exercise 6, conclude that I is generated by the 2×2 minors of M.

Notice that we didn't perform a single Gröbner basis calculation in this example, but yet we've proved that this set of minors is actually a Gröbner basis for I.

Exercise 12. Modify this exercise to prove that the defining equations for the d-uple embedding of $\mathbb{P}^1$ is given by the 2×2 minors of

$$\begin{pmatrix} x_1 & \ldots & x_d \\ x_2 & \ldots & x_{d+1} \end{pmatrix}$$

Exercise 13. Prove that the 2×2 minors of a $(m+1) \times (n+1)$ generic matrix generate a prime ideal as follows:

- Show that they vanish on the Segre embedding of $\mathbb{P}^m \times \mathbb{P}^n$. You can think of this via the ring map

$$k[x_{ij}] \to k[a_1, \ldots, a_{m+1}, b_1, \ldots, b_{m+1}]$$
$$x_{ij} \to a_i b_j.$$

  So show that the 2×2 minors are in the kernel of this map.

- What is the dimension of $\mathbb{P}^m \times \mathbb{P}^n$ and thus what is the codimension of I?

- The degree of $\mathbb{P}^m \times \mathbb{P}^n$ in this embedding is $\binom{n+m}{n}$, use this fact to prove that the minors for a full generating set of I.

- Perform a Hilbert Function calculation to show that this degree (i.e. multiplicity) is equal to $\binom{n+m}{n}$.

Exercise 14. Compute $\operatorname{gin} I$ for the following ideals of $\mathbb{C}[x, y, z]$

1. $I = (z^2)$
2. $I = (x^2, y^2)$
3. $I = (xy, xz)$
4. $I = (x^3, y^3)$.

You may just choose "random" changes of coordinates if you like. The Zariski-open condition is that the coefficients you encounter during S-pair calculations should never vanish because of your choice of coordinates.

4.3 Other Exercises

Exercise 15. Consider the ring $k[s^3, s^2t, st^2, t^3] \subset k[s,t]$. Various people might call this ring the 3rd Koszul subring of $k[s,t]$, but for me it will always be the twisted cubic, best thought of the imagine of the map $\mathbb{P}^1 \to \mathbb{P}^3$ defined by

$$[s:t] \to [s^3; s^2t; st^2; t^3].$$

Algebraically, this corresponds to a ring map

$$f : k[a,b,c,d] \to k[s,t], \quad f(a) = s^3, \ldots, f(d) = t^3.$$

1. Write down an appropriate ideal $J \subset k[a,b,c,d,s,t]$ such that $I = \ker f = J \cap k[a,b,c,d]$.
2. How could you compute this intersection?
3. How do the Betti numbers of J compare with those of I.

Exercise 16. ([1, Ex. 1.5.8a]) Using the Euclidean algorithm, prove that the following ideal of $k[x]$ is principal and find a generator:

$$I = (x^4 + x^2 + 1, x^4 - x^2 - 2x - 1, x^3 - 1).$$

Exercise 17. Prove that $k[x]$ is a principal ideal domain by proving that if I is generated by (f_α) then I is generated by the gcd of these generators.

Exercise 18. Prove Dickson's Lemma that any monomial ideal is finitely generated.

Exercise 19. Prove that on $k[x,y]$ there are only two monomial term orders that respect the grading. They are determined by whether $x > y$ or $y > x$.

Exercise 20.

- Verify that $\{x^2, y^2 + xy\}$ is a Gröbner basis with the Lex order induced by $y > x$.
- More generally prove that if $LT(g_1)$ and $LT(g_2)$ are relatively prime, then the remainder upon dividing $S(g_1, g_2)$ by $\{g_1, g_2\}$ is zero. Thus you can avoid checking some pairs. See Chapter 2 in [1] for more general results like this.

Exercise 21. Let $R = k[x_1, \ldots, x_n]$. Show that $HF_R = \frac{1}{(1-t)^n}$. This can be done in a few different ways via induction.

1. (Combinatorial) Prove that the number of monomials of degree d in R is $\binom{d+n-1}{d}$, and then check that these numbers satisfy an appropriate inductive relation.
2. (Functorial) Suppose that S and T are k-algebras. Prove that

   $$HS(S \otimes_k T) = HS(S) \cdot HS(T).$$

   The result follows by induction with $S = k[x_1, \ldots, x_{n-1}]$ and $T = k[x_n]$. (It's worthwhile to think through why this equality holds - what is an element of degree 10 of $S \otimes_k T$?)

Exercise 22.

1. If $R = k[x_1, \dots, x_n]$, show that the Hilbert series of $R(-j)$ is equal to $\frac{t^j}{(1-t)^n}$.

2. Conclude that if M is a finitely generated graded module with graded betti numbers β_{ij} then the Hilbert Series of M is equal to

$$HS(M) = \frac{\sum_{ij}(-1)^i \beta_{ij} t^j}{(1-t)^n}.$$

Exercise 23. Let f be a polynomial of degree d in $R = k[x_1, \dots, x_n]$. Compute $HS(R/(f))$. Your answer should only depend on the degree of f.

Exercise 24. Prove Corollary 2.19.

Exercise 25. Suppose that I is a homogeneous ideal and $G = \{g_1, \dots, g_r\} \subset I$ such that the

$$HS(R/I) = HS(R/(LT(g_1), \dots, LT(g_r)).$$

Prove that G is a Gröbner basis for I.

Exercise 26. Prove Proposition 3.6.

5 Some Open Questions

This is a short list of open questions that (maybe) could be solved with the using of degenerative techniques.

Question 5.1. (Buchsbaum-Eisenbud-Horrocks Rank Conjecture) Let I be a homogeneous ideal of height c in a polynomial ring R. Then

$$\beta_i(R/I) \geq \binom{c}{i}.$$

This is open even if $c = 5$, and as far as I know for $i = 2$.

Notice that when $i = 1$ this says that the minimal number of generators of an ideal is bounded below by the height of I, which is Krull's altitude theorem. Conceivably, one should be able to prove this using Gröbner bases and such an argument might extend for higher syzygies. This conjecture is known in several special cases, including the case when I monomial.

Question 5.2. (Sturmfels) Suppose that A is a finite collection of vectors in $\mathbb{Z}^d$. Then the ring $R = k[\mathbf{x}^a \ |a \in A]$ is called a toric ring. For instance, if A consists of the columns of

$$\begin{pmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

then R is the coordinate ring of the twisted cubic in $\mathbb{P}^3$. If additionally, it is normal, then a result of Hochster proves that R is Cohen-Macaulay. Suppose that $n = |A|$ If we write $R = k[x_1, \ldots, x_n]/I_A$ then I_A is a binomial ideal generated in degree at most d (see [7, Chapter 13].) Sturmfels has asked whether every such ideal I_A necessarily has an initial ideal that is Cohen-Macaulay.

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