1 Homework 1 - Due Friday September 1st at 5pm

Welcome to our first week of class! The goal of this homework is to get you familiar with Gradescope software, scanning your work, and uploading it.

- You will be doing this for each homework assignment
- Although I personally love physical paper, I am using Gradescope because it allows me to give you prompt and detailed feedback on your writing. You will be writing **a lot** in this class!

Instructions: Go to Gradescope.com and create an account using your USD email address. Once you have logged in click "+Add a course" and use the code: 2KNE46 which should add Math 360 to your dashboard. You should see this first assignment there. HW assignments will always be posted on Gradescope as well as in the Google drive folder.

On a plain sheet of paper (preferred) or lined paper, please write a solution to the following problem. If you have a tablet you can use that to write (it might be easier to produce a PDF that way.)

On this problem I'm mostly just looking to hear your ideas and to let you practice expressing yourself. On every assignment you will get full credit as long as you submit honest work that indicates you made a good-faith effort. If you would like to work with other students or come talk to me about the problems that is great and encouraged. In upper division math classes, homework problems might feel challenging (and even frustrating) but that is part of the process. By working through this "productive struggle" we learn and grow, which will deepen our understanding and improve our skills on quizzes and writing assignments. Do not post your homework questions to online homework help sites or copy solutions that you find on the internet - doing so is a violation of the USD Honor Code. Such violations will be reported and can result in serious consequences including failing the course.

Problem 1: A student makes the claim that "If x is a real number then $x^2 \geq x$." Is this claim true or false? Explain why or why not? If not, can you specify a set of real numbers x for which this claim would be true? Draw the graphs of $y = x^2$ and $y = x$ on the same axis. Can you connect this picture with your argument? Do your scratch work separately and when you are ready to write your submission, use a clean sheet of paper, complete sentences, legible handwriting and clear pictures.

Now you will need to scan your work **as a PDF**, (you can use any app you want, but some suggestions are on the attached pages) and upload this to Gradescope. This will show you what the interface is like. That's it for today - see you in class on Friday!

If you want to get started on Homework 2, due on Tuesday, feel free!

2 Homework Due Tuesday September 5th

1. In the coming weeks we'll want to start thinking about mathematical statements. For now, I'd like you to just think and write briefly your thoughts on the following:

   (a) Let P be the statement "It is raining now." What do you think is the opposite of P? (Sometimes people say "negation" to mean opposite)

   (b) Let Q be the statement "It rains every day." Is the opposite of Q "Some days it doesn't rain." or "It never rains." ?

   (c) Is the statement "$x \geq 0$" true or false (or neither - why?)

   (d) Is the statement "if $x = a^2$ for some real number a, then $x \geq 0$." true or false?

   (e) (On pants) Do you think there is a difference between the following two statements? Which do you think are true?

   (1) "There is a pair of pants that fits every person in the world."

   (2) "For each person in the world, there is a pair of pants that fits that person."

   (f) (On numbers) Is there a difference between these two statements? Which do you think are true?

   (1) "There is a natural number that is bigger than every real number."

   (2) "For every real number, there is a natural number bigger than it."

   For these last two parts, don't worry so much about what the "correct" answer is. You'll get credit for whatever you write. But I'd like you to think carefully about these statements and try to identify what your intuition says, and where you might find difficulty.

2. Consider the statements:

   A. "The sum of the first n squares is equal to $\frac{1}{6}n(n+1)(2n+1)$" (a square is a number of the form k^2 where k is a natural number)

   B. "$7^n - 2^n$ is divisible by 5 for all natural numbers n."

   C. "The sum $3 + 11 + \cdots + (8n - 5) = 4n^2 - n$ for all positive integers n.

   D. "The product of any finite number of odd numbers is odd"

   E. "If n functions $f_1, \ldots, f_n$ are all $1-1$ then so is the composition $f_n \circ f_{n-1} \circ \cdots \circ f_1$." (You might not remember what 1-1 means, that's ok - you don't need it to state P_1, P_n, etc. But you might want to go back and review 260 / 262 material about 1-1 and onto functions. This will come up later in class.)

   These are all examples of statements that can be proven by induction.

   - For each of parts ABCDE write down what statements P_n, P_1, P_4, P_{n+1} would be. You may **not** use dot dot dot ... notation for P_1 or P_4, please write everything out. For P_n and P_{n+1} it's ok to use dots if you need them.
   - Choose **one** of ABCDE and write a complete induction proof. Remember this is your chance to practice writing proofs and I'll be grading for feedback (meaning if you turn in a complete assignment that shows earnest work you'll get full points), so feel free to push yourself and challenge yourself to solve a tougher problem. This will help you greatly in your writing. If you want to see additional practice, check out my Youtube Channel which has

helpful videos! `https://www.youtube.com/watch?v=T1tV95b3U4s&list=PLsERF3FXCuK1HO_BpCI6kxi0RMZz1UWBA&index=2&t=0s`

3. One thing that is new in this class is that we have to deal with **inequalities** and **absolute value.** In almost all cases, when you are trying to show, say that BLAH < FOO, you will do this by starting with BLAH and connecting a chain of inequalities $<, \leq, =$ together and arrive at FOO. For example, here is a correct proof that my age (38) is less than 101.

$$\text{my age} = 38 < 40 \leq 25 + 25 = 25(2) \leq 25(4) = 100 < 101$$

   Read each step of this carefully. There are seven (!!) different statements in that chain, starting with the first = and ending with the last <. All of them are correct, and since the inequalities all go in the "same direction", we can conclude that "my age < 101". (Technically this follows from what we'll learn is called the transitive property of <.)

   Now here is a "flawed proof" that my age is < 26. As much as I'd like for this to be the case, it is not correct. The reason is that **even though each of the 7 individual statements are true** because the inequalities do not flow in the same direction, we cannot conclude that I am < 26.

$$\text{my age} = 38 < 40 \leq 25 + 25 = 25(2) > 25(1) = 25 < 26.$$

   Of the seven (in)equalities only one of them is in the "wrong" direction, but this seemingly small mistake has disastrous effects on the anti-aging proof. This just goes to show you that we need to be super careful when working with inequalities.

   A) Write a complete induction proof of the statement "If $n \in \mathbb{N}$ and $n \geq 2$ then $n^2 \geq n + 1$." This should be similar to the one we did in class. Under each = or $\geq$ symbol, please write an explanation. You might want to list the steps on different lines for more space like we did in class.

   B) (Optional, but recommended): Write an induction proof that if $n \geq 4$ then $n! \geq n^2$.

4. A) Draw a number line (a big one - spanning your whole page) and carefully mark the numbers from -5 to 5. I want you to **shade in** the following region:

   "The set of numbers x such that the distance from x to 1 is at most (meaning $\leq$) 2."

   (so for example make sure you shade in the number 0.3 because its distance to 1 is only 0.7. But don't shade in -4 since its distance to 1 is 5 which is too big.) Your answer should be an interval. Where is it centered? What is its radius?

   B) Go to Desmos and explore what happens as you type things of the form:

$$|x - a| < b, \qquad or \qquad |x - a| \leq b.$$

   What happens if b is negative? Why? What would a and b be to get the interval you shaded in the previous problem? Complete the following sentence:

   "$|x - a| < b$ means that the distance between ______ and ______ is less than ______."

2 Homework Due Tuesday September 5th

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   (so for example make sure you shade in the number 0.3 because its distance to 1 is only 0.7. But don't shade in -4 since its distance to 1 is 5 which is too big.) Your answer should be an interval. Where is it centered? What is its radius?

   B) Go to Desmos and explore what happens as you type things of the form:

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   What happens if b is negative? Why? What would a and b be to get the interval you shaded in the previous problem? Complete the following sentence:

   "$|x - a| < b$ means that the distance between ______ and ______ is less than ______."

HW2 selected solutions and comments

1. In the coming weeks we'll want to start thinking about mathematical statements. For now, I'd like you to just think and write briefly your thoughts on the following:

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3 Homework Due Friday Sept 8 - 5pm

1. The following statement is FALSE. Convince yourself why and provide a **counterexample**. Note this is saying that in general we can't just "take the squareroot of both sides of an inequality."

   Statement: If x and a are real numbers and $x^2 < a^2$ then $x < a$.

2. Ross 3.3 and 3.4 ask you to prove 4 different "basic properties" of the real numbers. Write these proofs carefully and neatly, explaining which axioms you use for each step.

3. It turns out that there is NOT a way to turn the set $\mathbb{C}$ of complex numbers into an ordered field. In this homework problem you will see why. Recall that $\mathbb{C}$ is the set of all numbers $a + bi$ where $a, b \in \mathbb{R}$ and $i^2 = -1$.

   (a) Explain why if $\mathbb{C}$ is an ordered field, it must be true that $0 < 1$. (Your answer should be a specific Theorem from the book).

   (b) Explain why if $\mathbb{C}$ is an ordered field, $0 \leq (i)^2$ (Your answer should be a specific Theorem from the book).

   (c) Using parts a) and b) explain why this shows that $\mathbb{C}$ cannot be an ordered field.

4. A very common type of statement we will encounter in analysis is:

   Given: $a < c$ for every $c > 5$

   Your job on this homework is to just understand what this statement is saying and draw some pictures. Here are some questions to help your investigation.

   - You are given some information about a: Are you told that $a < 10$? that $a < 2$? That $a < 5$? That $a < 5.1$ that $a < 5.00001$?
   - Draw a number line and mark in the number 5 and think about visually where the number c could be.
   - You probably have realized that it's possible that c could be 6. Does that mean that a could be 5.5? Actually NO! Make sure you understand this. This might be worth talking through with a friend. Yes it is 100% true that c could be 6, and yes 5.5 is indeed less than 6. However, $a = 5.5$ does not satisfy the **given** statement.
   - Write down a **conjecture** of what you think you can say about a. Your conjecture will likely be something of the form: Under the given condition, we can conclude a [some condition on a that will involve an inequality $<$ or $>$ or $\geq$ or $\leq$.
   - On Monday we'll talk about how to prove such a statement. For fun, you might want to try it now. Hint: we will use proof by contradiction.

(Extra Credit) 

Here's something really cool! Suppose you think of a natural number between 1 and a $1,000,000$ (a million). What are the chances do you think, that the two numbers are **relatively prime** (meaning that they don't have any factors in common other than 1)? After you have your guess, head to the lab (i.e. your computer) and do some experimentation. If you know some coding, this is a good chance to practice writing some short code. If you don't, you can use Google sheets or Excel and the GCD function. (Greatest Common Divisor). The idea is to generate lots of random pairs of numbers, say $a = 36$ and $b = 100$ and have the computer check to see if they have a common factor. (In this case they do - a factor of 4). After your search, see what percentage of those pairs have $GCD(a,b) = 1$ (the ones with no factor in common other than 1). You should get some percentage. What is this mystery percentage? See if you can come up with an approximation and then either write up your findings in a report and email it to me, or come by office hours and we can talk about it. Please don't google this one, there's a really cool punchline to this problem and the reward is greater if you've worked on it a little bit before the "reveal." These extra credit opportunities have no due date, so feel free to come back to them whenever in the semester.

What's on the Quiz Friday?

Our first quiz will cover the basic **concepts** we have learned (e.g. is the set $\mathbb{Q}$ a field? Is it an ordered field?) It will also cover problems from HW2, particularly questions about **distance** (remember Desmos?) and about induction. I will not ask you to write a full induction proof, but I might ask you what statement P_6 would be, or P_{n+1}. I might also ask you questions about the logic of how induction works (e.g. what is the role of going from one step to the [Thank u], next one.) Just for extra practice with **distance**, make sure you are comfortable converting sentences like "The distance between Joe and x is less than 10:

$$|Joe - x| < 10.$$

Try a few yourself.

- The distance between 9 and x is at most the distance between 7 and y.
- The distance between $(x + 4)$ and 3 is less than b. (Simplify the inside of the absolute value)
- The distance between $c + 8$ and $x - 7$ is less than r. (Simplify the inside of the absolute value)
- If $|x + 3| \leq 12$ then what is the largest x could be? Can you draw a number line to help you?
- If $|x + 3| < 10$ rewrite this as a "the distance between... " statement and draw a number line indicating the region.

4 Homework 4 - Due 5pm on Tuesday September 12th

1. Suppose that you wanted to show that quantity n^3+5n+3 at some point will be at least 1000 and stay that way for all larger n. One way would be to use Calculus, including solving this cubic equation, looking at increasing/decreasing intervals / maybe limits. This would require lots of proofs, messy equations etc. A **better** way for this point in the class it to do something like this:

   "Notice that $n^3 + 5n + 3 \geq 5n$. So that means that as soon as $n \geq 200$ then

   $$n^3 + 5n + 3 \geq 5n \geq 1000.''$$

   This isn't the only way to do this, but it works, and for this problem that's all that matters. Here's another strategy:

   "Notice that $n^3 + 5n + 3 \geq n^3 \geq n^2 \geq n$ so if $n \geq 1000$ then $n^3 + 5n + 3$ will be at least 1000 as well."

   > Notice: So far in this class we have not talked about square roots, or cube roots. There's a reason for this - if you can find a way to prove, e.g. that 2 has a square root in the real numbers (using the 15 axioms of the real numbers) please let me know. It's possible, but takes about a page of messy equations. The point of these above arguments is that they are **very simple** and don't require any sophisticated tools, just stuff like $n^2 \geq n$ which follows by say, multiplying the equation $n \geq 1$ by the positive number n.

   make sure that these arguments make sense to you and feel good. Please come see me in office hours or schedule an appointment if not.

   Here's a proof that is NOT valid. Suppose you wanted to prove that $n^2 - 5000n + 3$ is eventually greater than 1000.

   "Notice that $n^2 - 5000n + 3 \geq n^2 + 3$ OH SNAP! No need to read further - this statement is not true! Do you see what we did? That $-5000n$ on the left side is gonna make it actually smaller. And that will not help us in our quest to show this number is bigger than 1000.

   When we make a mistake like that, it's going to mess up anything that follows, so it doesn't even matter what came next. Like in a **computer program**, once something is incorrect, we should stop reading/writing. Here's a correct way to show the claim:

   "Notice that $n^2 - 5000n + 3 \geq n^2 - 5000n = n(n - 5000)$. So now if $n \geq 5002$ then the last factor will be at least 2, and the first factor is at least 5002, so if $n \geq 5002$,

   $$n^2 - 5000n + 3 \geq n^2 - 5000n = n(n - 5000) \geq 5002 \cdot 2 = 10004 \geq 1000."$$

   I hope this is fun for you - we've got one more before your problem.

   What if it happens that there are more than one term with a minus sign? Like say I wanted to show that $n^5 - 10n^3 - 20n^2 + n - 4$ was eventually greater than 1000 at some point. Well look at this

   > **Critical Observation:** For any natural number n: $n \leq n^2 \leq n^3 \leq n^4$. (This makes sense)
   >
   > and this implies that if we put negative signs, then e.g. $-n^3 \leq -n^2$ (like $-10^3 \leq -10^2$). So if we want to write inequalities, we can say $-10n^2 \geq -10n^3$. (the number on the right is more negative)

So what we can do is say:

"Notice

$$n^5 - 10n^3 - 20n^2 + n - 4 \geq n^5 - 10n^3 - 20n^2 - 4 \geq n^5 - 10n^4 - 20n^4 - 4n^4 = n^4(n-34) \geq n - 34$$

which will be at least 1000 if $n \geq 1034$.

A) for your homework, explain what changed in the first two inequalities and WHY they are each true. You should have a different answer for each. For this homework, what's most important is that I want everyone to just analyze those two inequalities and make sure you see why they are true for all $n \in \mathbb{N}$. In fact they both could be $>$ rather than $\geq$.

B) Use these ideas to prove that $n^3 - 10n^2 - 30n + 6$ is eventually at least 500 and stays that way for all greater n. Your proof should give a concrete value like "If $n \geq ...$".

2. This problem is about using the triangle inequality and understanding absolute value. Remember that knowing something like $|\clubsuit| < 5$ is the same as saying "$\clubsuit$ is between -5 and 5" i.e. $-5 < \clubsuit < 5$.
   - By converting to a double inequality, explain why if $|x-5| < 1$ then x is between 4 and 6.
   - Recall that the triangle inequality says that $|x+y| \leq |x| + |y|$. This is true no matter what you choose x and y to be. Use this to prove that

     $$|a-b| \leq |a| + |b|.$$

     (In your explanation, explain what you chose x and y to be. Hint: $a - b = a + (-b)$.)
   - Now use the triangle inequality to prove that $|a-b| \geq |a| - |b|$. Hint: (As a first step, rewrite this so the inequality is facing $\leq$ and that you have two terms on the right side. Clearly state in your explanation what x and y you are choosing.
   - Suppose you are given that $|x-3| < 1/2$ and $|y-3| < 1/2$ then $|x-y| < 1$. Use this information to prove that $|x-y| \leq 1$. Do this by writing: $|x-y| = |(x-3)-(y-3)| = ...$ and then using the triangle inequality. Be clear and careful with your steps.
   - Explain why if $|x-a| < 1/2$ and $|y-b| < 1/2$ then $|x+y-(a+b)| < 5$.

3. Suppose that $x, x_0 \in \mathbb{R}$ and $x_0 > 0$ and $|x - x_0| < \frac{x_0}{3}$.

   (a) Draw a number line depicting this situation. Make it large and clearly show the relative position of $0, x_0$ and include a shaded range for where x could be. (Label the endpoints - this will require adding fractions.)

   (b) Explain why your picture shows pictorally that $2x_0/3 < x < 4x_0/3$.

4. Let $a, b \in \mathbb{R}$. Suppose that X is a set of real numbers such that $\forall x \in X$, $a \leq x \leq b$. Prove that there exists a positive real number c so that $\forall x \in X$, $|x| \leq c$. (With this problem and most problems in this class draw a number line!) A full solution to this problem requires an explicit description of why you know c exists. The main way to do this is to give a formula for c in terms of a and b. Your formula might include words, like max/min. Be very specific.

5. **Book problem:** For the following sets from Ross Exercise 4.1 determine if the set is **bounded above**. If so give **an upper bound** and identify the **supremum** (no proof required). Is the supremum a **maximum**? Determine if the set is bounded below and if so give a **lower bound** and identify the **infimum** (no proof required). Is the supremum a **minimum**? You can simply make a table in your homework if you like.

(a) Sets (h), (i), (j), (k), (r), (w)

6. Let S be a non-empty subset of $\mathbb{R}$.

   (a) Prove that $\inf S \leq \sup S$. (Hint: Yes, this should be short proof.)

   (b) If $\inf S = \sup S$, what can be said about S? (No need to prove, just say what this means about S.)

   (c) Now suppose that T is another subset of $\mathbb{R}$ with $S \subset T$. (This means that S is a subset of T). Which is true:

$$\inf S \leq \inf T \qquad \text{or} \qquad \inf T \leq \inf S$$

   And which of these is true:

$$\sup S \leq \sup T \qquad \text{or} \qquad \sup T \leq \sup S$$

   Illustrate with examples to explain how you know this is true. (You are not required to write a proof for this problem).

7. Suppose that A is a nonempty set of real numbers and that $B = \{-a \mid a \in A\}$. Prove that if A is bounded below then B is bounded above. Hint: start your proof with: "Suppose that A is bounded below by the number m. Then ... " Your proof should indicate a specific upper bound for B along with a proof.

5 Homework 5 - Due Friday September 15th at 5pm

1. Prove that if $a > 0$ then there is an $n \in \mathbb{N}$ such that $\frac{1}{n} < a < n$.
   - Be careful in your proof.
   - For instance, here is an INCORRECT proof. "By AP there is an integer n such that $a < n$. Using the AP again, we see there is an integer n such that $\frac{1}{n} < a$. Putting these together we see that
   $$\frac{1}{n} < a < n.$$
   - The mistake in this proof is that when we use the AP we can get different numbers, e.g. if $a = 0.21$ then maybe the AP will return $a < 1$ and $\frac{1}{5} < a$. So if we put them together we get
   $$\frac{1}{5} < a < 1.$$
   which is NOT of the form $1/n < a < n$. Your job in this problem is to figure out how to fix the proof. Hint: What would your value of n be in this case? $n = 1$? $n = 5$? some other number?
   - As a first step, I recommend using different letters to represent the two numbers you get from the AP, and then think about how you can somehow combine these to get the number n that works on both sides. The words max and/or min might be helpful for this problem.
2. Prove inf of $S = \{\frac{1}{2n} : n \in \mathbf{N}\} = 0$. (You'll need to show two things 1.) that 0 is a lower bound, and 2.) that any bigger number is NOT a lower bound. Your proof should use the Archimedean Principle)
3. Prove that $\sup\{3 - \frac{1}{2n+1}\} = 3$.
4. Using the Archimedean Principle show that the set $S = \{n^2 + 3n + 5,\ n \in \mathbb{N}\}$ is unbounded. Your proof should start as follows:

   Proof: I will show that the set S is unbounded, by showing that no number M is an upper bound. Let $M \in \mathbb{R}$. I will show that M is NOT an upper bound by finding an element of S that is $> M$.

   [Now is where you have to use the AP like we did in class. Be specific and careful with your proof. This is an example of a type of proof I might ask on a quiz. Your proof doesn't need to be very long.]

What's on the quiz? The quiz this week will have some conceptual questions asking about the theory of the real numbers. What does the completeness axiom say? Is $\mathbb{Q}$ a complete ordered field (no). Does every subset of the real numbers have a supremum? I might also ask a homework question related to one of the homework questions from this week. For instance, can you find the sup / inf of a set like described in the book HW. Can you draw a number line to represent data from inequalities / absolute value? Can you use the triangle inequality? Can you write a simple sup/inf proof? Can you use the AP and write clear correct statements? Can you state the completeness axiom?

EC (Lots of Extra Credit) - This problem might take the whole semester) Calculate (with proof) the number:
$$\sup\{\sin n \mid n \in \mathbb{N}\}.$$
You can use whatever definition of $\sin n$ you like, but what you really need to know is where $\sin n$ has its peaks. You may use without proof that π is irrational. And yes, you will need this fact.

As a start, try some experiments and make some conjectures / guess and then come talk to me / send me your report.

6 Homework Due Tuesday September 19 - 5pm

1. Let a be a real number. Prove that

$$\sup\{r \in \mathbb{Q} \ : \ r < a\} = a.$$

   Hint: You can use the Denseness of the Rational Numbers. Be precise and clear with your writing. This is a good example of a short proof that could be on a quiz.

2. Below you will read two proofs of a claim. The first one is not correct and has a serious error. Identify the error and make sure you understand why it is not correct. Write a few sentences explaining **why** the proof doesn't work. The second proof is valid. Note: the error is subtle, and the second proof doesn't really call explicit attention to this detail. Read very critically and you've got this!

   Claim: The set $S = \{n^2 - 3n + 1 \ : \ n \in \mathbb{N}\}$ is not bounded above.

   Flawed Proof: We will show S is not bounded above. Let $M \in \mathbb{R}$. We will show that M is not an upper bound. By the AP $\exists n \in \mathbb{N}$ such that $n > M + 3$. Then

$$n^2 - 3n + 1 \geq n^2 - 3n = n(n-3) \geq 1 \cdot (n-3) > (M+3) - 3 = M.$$

   Hence we have found an element of S that it greater than M. So M is not an upper bound.

   Correct Proof: We will show S is not bounded above. Note that $1 \in S$ (by setting $n = 3$). Thus if M is an upper bound then $M \geq 1$. Let $M \geq 1$ be a real number. We will show it is not an upper bound. Indeed, by the AP $\exists n \in \mathbb{N}$ such that $n > M + 5$. Then

$$n^2 - 3n + 1 \geq n^2 - 3n = n(n-3) \geq (M+5)(M+2) \geq M^2 \geq M.$$

   Hence we have found an element of S that it greater than M. So M is not an upper bound. Thus S is not bounded above and thus not bounded.

3. In this problem you may use the fact that $\sqrt{2}$ exists (we haven't proven this, but we could...) and that it is irrational (we **have** proven this). You may also use the fact that if n is a natural number then $\sqrt{2}/n$ is irrational.

   (a) Write down a sequence of irrational numbers that converges to a rational number

   (b) Write down a sequence of irrational numbers that converges to an irrational number (hint: what is the simplest possible sequence you can come up with?)

4. Take a look at the book problems in Ross Section 7. These should feel like Calculus 1 problems. Choose at least one part from 7.1, 7.2. Choose at least 4 parts from 7.3, and try at least one from 7.5. Hint for 7.5 (first test with a calculator to see what the limit might be and then "multiply by the conjugate".) Choose options so you can check your work in the back of the book.

7 Homework Due September 22nd - 5pm

0. Problem 0 is optional and you do not need to submit it. **Problems 1, 2, 3 need to be submitted** Problem 0 gets you to start thinking about sequences and numbers and how they can fit together.

   (a) In writing a proof about $\sqrt{n+\sqrt{n+3}}$ where $n \in \mathbb{N}$, which would be correct? (Circle all that apply - and you should circle two)

   $$\sqrt{n+\sqrt{n+3}} \leq \sqrt{n} \qquad \frac{1}{\sqrt{n+\sqrt{n+3}}} \leq \frac{1}{\sqrt{n}}$$

   $$\sqrt{n+\sqrt{n+3}} \geq \sqrt{n} \qquad \frac{1}{\sqrt{n+\sqrt{n+3}}} \geq \frac{1}{\sqrt{n}}$$

   (b) Going further, how could you compare $\frac{1}{\sqrt{n}}$ with $\frac{1}{n}$?

   (c) Can you define a sequence of rational numbers that converges to an irrational number? (feel free to use decimal expansions) (This is for intuition, not rigorous proof)

   (d) Can you write a formula $s_n =??$ for a sequence of rational numbers that converges to a rational number?

   (e) Draw on a number line the region described by $|s_n - 3| < .01$. (Your picture should be an interval of radius .01 centered at 3. This represents the zone where s_n could be and satisfy this inequality.)

   (f) Give an example of a sequence that does not converge, but that does satisfy the following condition (note this condition is NOT the definition of convergence)

   $$\exists \epsilon > 0 : \exists N : \forall n > N, \; |s_n - 1| < \epsilon.$$

   (Your answer should clearly give an s_n, ϵ, and N. It's ok if you want to define s_n in by listing numbers and using ....

   (g) Suppose you want to show that a sequence s_n converges to 4.6. Which of the following would be true?

      i. There is at least one s_n in the range: $4.5 < s_n < 4.7$

      ii. There are infinitely many s_n in the range: $4.5 < s_n < 4.7$

      iii. There are infinitely many s_n in the range: $4.5 < s_n < 4.8$

      iv. $s_{1000000}$ is in the range $4.5 < s_{1000000} < 4.7$

      v. On the number line if you shade a small symmetric interval around 4.6 then all terms of s_n will be in this shaded region.

      vi. On the number line if you shade a small symmetric interval around 4.6 then after some point N, all terms of s_n will be in this shaded region. The number N depends on how small the shaded interval is.

      vii. On the number line if you shade a small symmetric interval around 4.6 then after some point N, all terms of s_n will be in this shaded region. The number N can be chosen independently of the size of the shaded interval.

1. At this point in the class your limit proofs will use the definition and Archimedean Principle. Please follow the format we've been doing in class. The following problems are all reasonable problems that I recommend working on in advance of Friday's quiz. **Please submit 1 of these on your homework for feedback. They are listed in rough order of difficulty.**

   (a) Find with proof the limit of the sequence defined by $a_n = \frac{(-1)^n}{n}$.

   (b) Find with proof the limit of the sequence defined by $a_n = (2n-1)/(3n-10)$

   (c) Find with proof the limit of the sequence defined by $a_n = n/(n^3+4)$ (Hint: Make sure you remember your goal is to make this very very small. If you simplify as $\leq \frac{n}{n}$ this will be true, but not helpful, because you'll not be able to show this is arbitrarily small. Instead, why not compare with $\frac{n}{n^3}$ and see what that gets you. These strategies that you are developing on homework will come in handy throughout the course.

   (d) Find with proof the limit of the sequence

$$s_n = \frac{4}{\sqrt{n+\sqrt{n+3}}}$$

   (e) Prove that $\lim \frac{\sin n}{n} = 0$. Note that we haven't defined $\sin x$ yet, but for this exercise use whatever definition of the sine wave that you know, e.g. that $-1 \leq \sin x \leq 1$. You'll see that you need very little about $\sin n$ to solve this problem.

   (f) Prove that if $c \in \mathbb{R}$ then the sequence defined by $a_n = c$ (the constant sequence $\{c, c, c, c, \ldots\}$ converges to the limit c. (This is to test your understanding of the definition. Sometimes the simplest sequences cause the most trouble)

2. Suppose that a sequence satisfies the following condition. What can you say about the sequence? Must it converge? Diverge? Can't tell? Be as specific as you can. Notice it says $\epsilon \geq 0$ this should be very helpful.

$$\forall \epsilon \geq 0, \exists N : \forall n > N, \quad |s_n - 6| < \epsilon.$$

3. Prove that the limit of the sequence $a_n = 1/(2^n)$ is 0. (Hint: You can use what we proved on Day 2 that if $n \geq 5$ then $2^n \geq n^2$. This is a very good problem to test your use of the AP. If you need in your proof that e.g. $N \geq 5$ then you can write something like by AP, take $N > \max(5, ??)$)

8 Homework Due September 26th - 5pm

0. (Optional)

   (a) Find two sequences such that $a_n < b_n$ for all n but where $\lim a_n = \lim b_n$.

   (b) Find two **bounded** sequences x_n and y_n that do NOT converge, but such that x_n/y_n DOES converge.

1. Carefully write up proof **using the Limit Theorems** from Day 10, that

$$s_n = \frac{4n^3 - 17n^2 + 3}{6n^3 - 45}$$

   converges to 2/3.

   You may use all the theorems in the box and extend Number 4 to say that a_n/b_n converges to a/b with the given hypotheses.

2. This problem has three "shorter" theoretical proofs.

   (a) Prove that the constant sequence $s_n = c$ converges to c. (Hint: your proof should be very short, but make sure you say what November (capital N) can be.).

   (b) In class on Monday we will prove that if $a_n \to a$ and $b_n \to b$ then $a_n + b_n \to a + b$. Practice writing this proof a few times and make sure you **understand** it.

   (c) Try to prove that if $a_n \to a$ and $c \in \mathbb{R}$ then $(c \cdot a_n) \to ca$. Your proof should be similar to the previous problem. If you get stuck, re-read this section of the book. This is one of the theorems proved in that section. But remember your goal is to **understand and internalize the proof.**. I might ask a problem on Friday's quiz that asks you to prove something similar.

3. Suppose that (t_n) is a bounded sequence, i.e. there is some number M such that $-M \leq t_n \leq M$ for all n. Let s_n be a sequence such that $\lim s_n = 0$. Prove that $\lim(s_n t_n) = 0$.

4. Suppose that the sequence x_n converges to L. Let $k \in \mathbb{N}$ and define $y_n = x_{n+k}$. Prove that $y_n \to L$ as well.

> The above proofs should be starting to feel "not too bad." If not, please stop by office hours and let's practice putting a few on the board. The proofs below are considerably more difficult. You might want to get an early start on them.

9 Homework Due September 29th - 5pm

1. (a) Suppose that a_n is a sequence of **integers** that converges to L. Prove that L must be an integer. (In an earlier assignment you showed that it's possible for a sequence of rational numbers to converge to an irrational number (and vice versa). This this exercise you will show that IF a sequence of **integers** converges, then its limit must be an integer. This proof will likely feel hard - that's ok! But you will need to think about how you will argue this.)

   (b) In fact, show that if a_n and L are as above, then for sufficiently large n (i.e. there is some N so that for $n > N$) $a_n = L$. In other words, after some point a_n is a constant sequence.

2. Suppose that $z_n \to L$ and $L \neq 0$. Prove that there is an N such that if $n > N$ then $|z_n| > \frac{|L|}{2}$. (Draw a number line and pick a good choice for ϵ.

3. Suppose that x_n and y_n are sequences (that don't necessarily converge) but such that the sequence $x_n/y_n \to 1$. Suppose that x_n and y_n are bounded. Prove that $x_n - y_n \to 0$. Then give an example where both x_n and y_n are unbounded and $\frac{x_n}{y_n} \to 1$ and $x_n - y_n$ does not converge to 0.

8 Homework Due Tuesday October 10th at 5pm

O (Optional but Highly Recommended for Quiz Practice.)
Suppose that $a \leq 5$. It is NOT true $|a+10| < |5+10|$. Write down an explicit example that shows this.

O (Optional but Highly Recommended for Quiz Practice.)
Give an example of:

(a) Two divergent sequences a_n, b_n such that $a_n + b_n$ converges.

(b) Two divergent sequences a_n, b_n such that $a_n \cdot b_n$ converges.

(c) A sequence that is monotone but not Cauchy.

(d) A divergent sequence a_n such that $|a_n|$ converges.

(e) A sequence a_n with $6 < a_n < 7$ for all n such that a_n has a subsequence converging to 6 and a subsequence converging to 7.

(f) A sequence a_n such that for each $k \in \mathbb{N}$ there is a subsequence converging to $1/k$.

(g) A bounded sequence that does not converge to 4/9 but has a subsequence converging the 4/9.

(h) A sequence a_n with ALL THREE the following three properties:

1) It has a subsequence converging to 1

2) Another subsequence converging to 17

3) Another subsequence converging to e.

1. Prove using the Main Limit Theorems (no ϵ needed) that if a_n converges and $a_n + 3 \cdot b_n$ converges then b_n converges. **I don't want you to use ϵ here. I want you to cite the appropriate theorems.**

2. Using the definition of Cauchy (the one given in class and in the book) prove that IF x_n and y_n are Cauchy sequences then $x_n + y_n$ is a Cauchy sequence. (Get out the ϵ's.)

3. Let x_n and y_n be Cauchy sequences. Prove that $x_n y_n$ is a Cauchy sequence using the definition (Look back at the product rule proof for convergent sequences)

4. Here is a Theorem (you don't have to prove it):

> **Theorem:** Suppose that a_n is a sequence and that $a_n \neq 0$ for all n. Suppose that $L = \lim |\frac{a_{n+1}}{a_n}|$ exists. If $L < 1$ then a_n converges to 0.

Your homework: **USE** this Theorem to prove that the sequence $a_n = 5^n/n!$ converges to 0. (Don't overthink this - just check what the theorem says and write your answer in complete sentences. You should find yourself simplifying fractions and factorials.

(On a later homework I might ask you to prove this Theorem. If you want to think about why it's true, come talk to me in office hours.

O (Optional for practice) Prove that if $x_n \to L$ and $L > 0$ then there exist infinitely many n such that x_n is positive. (Get out the epsilons for this one!)

O (Optional Proof Portfolio Suggestion) - Ross 10.10

9 Homework Due Friday October 13th at 5pm

O (Optional) Which of the following sequences has a convergent subsequence? (You may use whatever theorems you like)

(a) $a_n = (-2)^n$

(b) $a_n = \begin{cases} 2^n & n \text{ is odd} \\ 5 & n \text{ is even} \end{cases}$

(c) $a_n = (-1)^n/n$

(d) $a_n = 2^{(-1)^n}$

(e) $a_n = \cos n$.

For each of the following sequences, determine how many different limits of subsequences there are. Justify your answer (briefly)

(a) $\{1 + (-1)^n\}$

(b) $\{\cos(n\pi/3)\}$

(c) $\{1, 1/2, 1, 1/2, 1/3, 1, 1/2, 1/3, 1/4, 1, 1/2, 1/3, 1/4, 1/5, 1, \ldots\}$

1. A) Explain why a Cauchy sequence cannot have an unbounded subsequence.

   B) Can an unbounded sequence have a Cauchy subsequence? If so, give an example.

2. In class we noticed that $\sqrt{n}$ is not Cauchy because it is not bounded. Now we'll prove that it doesn't obey the Cauchy condition: Prove that the series defined by $a_n = \sqrt{n}$ is not Cauchy, using the definition of Cauchy sequence. Hint and Template: Let $\epsilon = 1$ and show that no matter what N is, you can find $n, m \geq N$ such that $|\sqrt{n} - \sqrt{m}| \geq 1 = \epsilon$. Hint2: There are a lot of numbers - what's $\sqrt{1000000000^2} - \sqrt{1000000^2}$?

O (Optional) Make sure you can produce the proof we wrote in class: We showed that $a_n = \sqrt{n}$ still satisfies the property:

$$\forall \epsilon > 0, \ \exists N \ : \ \forall n \geq N, |a_{n+1} - a_n| < \epsilon. \tag{9.1}$$

Thus property (9.1) is **not** the same as the Cauchy property. Your proof should start "Let $\epsilon > 0$."

3. We now give a slightly different property for a sequence $\{a_n\}$:

$$\forall n \in \mathbb{N} \ \ |a_{n+1} - a_n| < 1/2^n. \tag{9.2}$$

   Prove that if a_n is **any sequence** that satisfies this property then a_n is a Cauchy sequence. Hint1: Try to see what you can say about $|a_{13} - a_{10}|$. You should use the triangle inequality and there should be 3 terms. Hint1.5: What happens now if you have $|a_{n+k} - a_n|$? Can you write a clear statement. Hint2: At some point in your proof you should see (maybe after factoring) an expression that looks like $1 + 1/2 + 1/4 + ... + 1/2^{blah}$). You proved in your homework last week that this sum will always be less than 2, because you showed that the whole infinite sum is equal to 2. (We also did this in class when we showed that e exists.) This is a tough proof, but you can do it! It's ok if you don't solve it this week, you can try again in future weeks and come to office hours.

O (Optional) - take a look at the sequence in Ross 11.2,11.3,11.4 parts a and b. These might appear on the quiz. Be able to answer questions similar to the above.

4. (Do this for Monday's class) I'd like you to submit your first problem for your proof portfolio. Bring your proof to class on Monday and after the definition quiz we will do some peer editing.

   Remember that this proof portfolio is for you to grow in your proof-writing, so please choose proofs that will **challenge you.** Ultimately, part of your grade on the proof portfolio will be that it shows growth in what you present.

   You can choose:

   (a) Any problem from previous homework provided that in the posted solutions I mentioned that this would be a good proof portfolio problem.

   (b) Write a proof in your your own words, (meaning don't just copy our proof from class) a proof that $(-1)^n$ does not converge. Your proof should still be rigorous, but it should give your own flavor to help convince the reader you really understand what is going on.

   (c) Prove that the sum of two Cauchy sequences is another Cauchy sequence by using the definition (this was HW on Tuesday)

   (d) Same question but for product.

   (e) Let S be a nonempty bounded set of real numbers. Suppose that $\sup S \notin S$. Prove that there exists an increasing sequence s_n of points in S such that $s_n \to \sup S$. In other words, an increasing sequence that converges to the supremum. Is this true if $\sup S \in S$? (no)

   (f) Use induction and the "product rule" for limits to show that if $x_n \to x$ then for all powers $p \in \mathbb{N}$, $(x_n)^p \to x^p$.

10 Homework Due Oct 3rd - 5pm

Theorem: If s_n is a sequence of positive real numbers that converges to L, then $\sqrt{s_n}$ converges to $\sqrt{L}$.

The above theorem will be helpful on the homework, you don't have to prove this theorem.

0. (Optional for extra practice). Suppose that a_n is a sequence of positive real numbers and $a_n < 10$ for all n. Prove that a_n does not converge to 15.

1. (Everybody's so creative!) In your Calculus Class you probably remember that you could have what were called **indeterminate forms**, like $0/0$ or $0 \cdot \infty$ and that that these expressions required lots of care. For instance if we look at the sequence $s_n = \dfrac{1/n}{3/n}$, the limit of the numerator and denominator are both 0, but the limit of s_n is 1/3. **In this problem you will be creative** and come up with all sorts of examples.

   Part 1: (0/0) Give examples of sequences a_n and b_n with $a_n \to 0$ and $b_n \to 0$ but such that

   (a) $\dfrac{a_n}{b_n} \to 5$.

   (b) $\dfrac{a_n}{b_n} \to 0$.

   (c) $\dfrac{a_n}{b_n} \to \infty$.

   (d) $\dfrac{a_n}{b_n} \to -\infty$.

   Part 2: $(0 \cdot \infty)$ Give examples of sequences s_n and t_n satisfying $\lim_{n\to\infty} s_n = 0$ and $\lim_{n\to\infty} t_n = \infty$ illustrating each of the following possibilities. Remember that our definition of $\lim a_n = \infty$ is that the terms are eventually all positive and $\lim(1/a_n) = 0$.

   (a) $\lim_{n\to\infty} s_n t_n = \infty$.

   (b) $\lim_{n\to\infty} s_n t_n = 7$.

   (c) $\lim_{n\to\infty} s_n t_n = 0$.

   (d) The sequence $s_n t_n$ is bounded but doesn't converge.

2. Suppose that t_n is a sequence defined as follows: $t_1 = 1$, and $t_{n+1} = \sqrt{t_n + 1}$.

   (a) Write down the first four terms of t_n using a calculator to get decimal approximations.

   (b) We will later show that t_n **converges**. Using this fact and the Main Limit Theorems (and the Theorem on top of this homework), prove that the limit is $L = \frac{1}{2}(1 + \sqrt{5})$. (Hint: look at the equation
   $$t_{n+1} = \sqrt{t_n + 1}$$
   and think about what the limit of each side should be. Write your answer in terms of L and then solve for L. You're a pro!

3. Similar to Problem 1, suppose that t_n is defined by $t_1 = 1$ and $t_{n+1} = \frac{t_n^2+2}{2t_n}$.

   (a) Write down the first four terms of t_n using a calculator to get decimal approximations.

   (b) We will later show that t_n **converges**. Using this fact, like in the previous problem, find the limit of t_n.

4. Similar to earlier problems, suppose that x_n is the sequence defined by $x_1 = 1$ and $x_{n+1} = 3x_n^2$.

(a) Write down the first four terms of x_n

(b) Show that IF $a = \lim x_n$ exists then $a = 1/3$ or $a = 0$.

(c) Does $\lim x_n$ exist? Why or why not?

(d) Discuss the appareant contradiction between b and c.

5. Ross 10.1

11 Homework Due Oct 6rd - 5pm

1. (a) Expand the following expression:

$$(1 - a)(1 + a + a^2 + a^3 + a^4).$$

Actually foil it out and see what you get. Use this to convince yourself that

$$1 + a + a^2 + \cdots + a^n = \frac{1 - a^{n+1}}{1 - a}$$

when $a \neq 1$.

(b) Use the above to calculate $\lim_{n\to\infty}(1 + a + \ldots + a^n)$ when $|a| < 1$. (Here I want you to use the formula above and the main limit theorems. You may need to cite part of Theorem 9.7 from the book.

(c) By using part (b). Calculate $\lim\limits_{n\to\infty}(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{3^n})$.

(d) As a gentle reminder - I hope you didn't google this problem or any problem on your homework. This problem is **very well within your ability and you can do it!** You might struggle at first to understand what it is asking, but I have broken it down into bite-sized pieces to help you. If you are in the habit of looking online for help - this will actually be counterproductive in this class. In this class you will be faced with new problems on homework and quizzes and the best help is to embrace the struggle. This problem is helping you **learn** how to deal with ..., by turning this sum into a sequence $\frac{1-a^{n+1}}{1-a}$ and then analyzing it with tools from the book.

Sometimes problems in this class will be challenging (I think Homework 9 contained our first very tough problem.) You are going to grow and learn the most by embracing that struggle and try to work through one step at a time.

2. The rest of the homework is about embracing the challenge of learning new things.

Take one of the problems from Homework 9 that you struggled with and try it fresh. See how it feels to start writing the proof again from scratch. What do you remember? What feels different? If you worked with friends on the problem last week, try switching up the roles - maybe if you were at the whiteboard, this time, let someone else try. Or if you worked alone, try working on this one with a group. Remember, the goal is for you to truly understand how every part of the proofs work in this class. This is a big ask, I realize - but I believe in you. I'll do all I can as a teacher to help, but ultimately learning comes from doing, and in this case that means letting our brains struggle with new concepts.

3. One of the most famous theorems about the real numbers is The Torero Mystery Hallowe'en Netflix Theorem. (This isn't the real name of this theorem of course - but in this problem I want you to try and understand what it is saying, and I want you to do this by thinking about the statement, (I believe in all of you - you can do it!) and I don't want you to find an online resource that explains it. Though surely that would make this process easier, it would prevent you from practicing one of the hardest parts of this course - reading complicated statements and trying to decipher them.) One of the most important skills you can cultivate in yourself is the skill of being able to read careful and complicated statements and understand what they are saying. **After this homework is due I'll tell you what the name of this theorem is. If at that point you are interested and want to read and write up a proof of this theorem that would be awesome! (And could count as 3 of your proofs in your proof portfolio!)** But for now we're not going to prove it, we're just going to work hard to understand what it says. It says the following:

   Let $n \geq 1$ be an integer and let x be a real number. Then there exists a rational number $\frac{p}{q}$ so that $1 \leq q \leq n$ and

$$|x - \frac{p}{q}| < \frac{1}{nq}.$$

   What the heck is this saying? Seriously - that's what I want to you figure out. And you have all the tools you need - you just need some paper. I want you to write a summary about this problem and submit it as part of your quiz on Friday. **Please write on paper or print out your work if you type or write on a tablet.** Basically I want you to write up some examples or explorations that you did while trying to understand this statement, what you learned, and a summary or what it is saying. The questions below are meant to help guide you, but you can work however you like. I want us to build our ability to read and really understand what things are saying. This is much more work than just passively reading. I want to help you do that explicitly for this problem. Your answer should convince me that you have worked through some of the different pieces of this statement and understand what it is (and isn't saying). The questions below are just an example of the types of questions that mathematicians (like you) should practice asking themselves when they encounter a new and technical statement. It's only by asking these things and struggling to understand what is going on that we can learn what a theorem is saying.

   - What does it say when $n = 1$ and $x = \pi$? Is the statement true in this case? What would the rational number p/q be?
   - What about if $n = 2$ and $x = \pi$? (Do you see what I'm doing?)
   - What about if $n = 3$ and $x = \pi$? (This is like debugging computer code by doing one simple case after another).
   - In all of the above are there multiple p/q that work? You will need a calculator for this. You might even want to use a spreadsheet.
   - In general, what is happening when $n = 1$ and x is just some number? What are the allowable denominators?
   - Hmm, what if $x = 3/10$? What does the theorem say? Could $p/q = 3/10$? (Be careful!!) Does it depend on what n is? What if $n = 4$ What are the possible values of p/q. What if $n = 9$? If $n = 9$ then what are the possible values of p/q?
   - If $x = a/b$ can I let $p/q = a/b$? Or does this depend on n?
   - If x is an integer, is this theorem clear or obvious? (The answer to this should be yes)
   - What if x is a rational number? (To me, Adam, I don't think so.)

- Suffice it to say, this theorem is very far from obvious in general. Is it obvious for general x with $n = 1$?
- This statement is all about "how closely can we approximate a number x by using rational numbers with a certain denominator."

14 Homework Due Tuesday October 17th at 5pm

1. I hope you're feeling proud of all you've accomplished - everyone's proofs are looking a lot better, and we're going to continue to grow. Last week I was going to prove that e was irrational, but ran out of time. But I was thinking and realized that this can be a great opportunity for you to do it! It's a reasonable argument and I'll scaffold it for you below.

   - We defined e as the limit of the sequence
   $$a_n = 1 + 1/1! + 1/2! + \ldots + 1/n!$$
   - In class we proved that a_n converges by the monotone convergence theorem, and in particular showed that
   $$2 < e < 3$$
   so that we already know at least that e is not an integer.
   - Your goal in this problem is to prove that e is irrational. I recommend you start by arguing by contradiction. "We will argue by contradiction. Suppose that $e = a/b$ for some integers a and b." Explain why you may assume already that $b \geq 2$. (Read above what we know about e so far. This might be important later, who knows?)
   - Now consider the following equation
   $$e = \frac{a}{b}$$
   $$1 + \frac{1}{1!} + \cdots + \frac{1}{b!} + \left(\frac{1}{(b+1)!} + \frac{1}{(b+2)!} + \cdots \right) = \frac{a}{b}$$
   on the left I just paused at the $1/b!$ term and then continued.
   - Now multiply both sides by $b!$. Explain why the number you get (the same number on both sides, $e \cdot b!$ has to be an integer. (Hint: look at the right hand side).
   - Now explain why the left hand side CANNOT be an integer. (part of it is an integer - but what about all those other terms. Can you prove they are less than 1? Remember there are infinitely many other terms. Can you use an inequality? Can you use the geometric series from HW 12, that $1 + p + p^2 + \cdots = \frac{1}{1-p}$ when $|p| < 1$.)
   - Really try to take care with this proof. This level of proof is something I will expect you be able to write accurately by the end of the course, say on the final exam (indeed, in Fall 2019 I put a version of this proof on the final exam), so work hard with your inequalities! You can put this in your proof portfolio. And I'd love to see your work along the way.

O. (Extra Credit if you Present During Office Hours) When I was learning analysis when I was in college this was one of the first problems that my professor gave me. (I was learning from him over the summer.) It's tough, but doable, and a perfect example of how we can "follow our nose" to find the right ϵ's and $N's$ for our proofs. Suppose that a_n is a sequence. Define a new sequence as
$$c_1 = a_1, \quad c_2 = \frac{a_1 + a_2}{2}, \quad \cdots, c_n = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

   (a) Prove that IF a_n converges to L then c_n also converges to L. Get out the epsilons! You should use that you can make $|a_n - L|$ as small as you like - but that's just where the fun gets started!

   (b) It is possible that c_n converges even when a_n does not. Let $a_n = (-1)^n$. Write down the first 10 terms for c_n in this case. (You should see a pattern). What does c_n converge to?

2. Let $f(x) = \sqrt{x-4}$ and $g(x) = \frac{1}{x}$. In Calculus you might have been asked "what is the domain of these functions." By this we usually meant "What is the largest subset of $\mathbb{R}$ on which these formulas would make sense." Do that below to get some practice.

   a. What is the domain of f?

   b. What is the domain of g?

   c. Give a formula for each of the following functions $f+g$, fg, $f \circ g$ and $g \circ f$?

   d. Is $f \circ g = g \circ f$?

   e. What are the domains of $f+g$, fg, $f \circ g$ and $g \circ f$?

3. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 3 - x & \text{if } x < 4 \\ x^2 & \text{if } x \geq 4 \end{cases}$$

   Prove that f is NOT continuous. Make sure you understand this proof. Ask yourself, "how would the proof change if I have changed it so that I used $\leq, >$ instead of $<, \geq$? Would my proof still work?" (Hint: your proof should definitely not still work, it will require some changes.)

4. We will work on these problem in breakout groups on Monday in class. In these three problems you will prove that functions are continuous (or not) using the criterion for continuity using sequences:

   (a) Prove that

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

   is not continuous at 0. (Hint: Can you find a sequence of numbers $x_n \to 0$ such that $f(x_n)$ does not converge to 0?)

   (b) Prove that

$$f(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

   is continuous at 0. (Hint: Begin your proof by saying "Suppose that $x_n \to 0$, then I will show that $f(x_n) \to f(0)$. ..." and then work out what you know about $f(x_n)$. Hint: you might need the squeeze theorem. Check out my youtube video on the squeeze theorem!)

   (c) Prove that the function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

   is not continuous at any point a. (Hint: there are two cases, one when a is rational and one when a is irrational)

15 Homework Due Friday October 20th at 5pm

As you saw in the previous homework, you can use the following facts:

1. In this problem you will prove two important facts that you've implicitly been using. The first is very easy, the second is a little harder.

   (a) Let x be an irrational number. Explain, in a few sentences, why the sequence $a_n = x + 1/n$ is a sequence of **irrational** numbers that converges to x. (Hint: if a_n were rational, what would happen if you solved for x?) Write your explanation clearly.

   (b) Let $x \in \mathbb{Q}$. Explain why the sequence $a_n = x + \sqrt{2}/n$ is a sequence of irrational numbers that converges to x.

   > These are good sequences to know about. For instance, say you need a sequence of irrational numbers that are all less than 4 that converge to 4. How could you modify your work in parts a or b?

   (c) Finally, sometimes you'll need to find a sequence of rational numbers that converges to a given number. You can argue as follows: Let $x \in \mathbb{R}$. Then by the denseness of the rationals, we know that there is a rational number in the interval $(x - 1, x)$. Let's call that number a_1. Similarly there is some rational number number in the interval $(x - \frac{1}{2}, x)$. Let's call that number a_2. Continuing this way, write (using complete sentences) a description for how to find a sequence a_n that converges to x. Draw a number line to help you visualize what you are doing.

   > These ideas will be useful for the next problem.

2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $f(q) = 0$ for all $q \in \mathbb{Q}$, then prove that in fact $f(x) = 0$ for all $x \in \mathbb{R}$.

   Write your proof using the first definition of continuity that we gave (the one using sequences.) Hint: How do we start this proof?

   "Let $x \in \mathbb{R}$ then we want to show $f(x) = 0$...." [Now use some of the ideas from the previous problem to construct sequences that help you with what you want to show!]

3. You just proved an important theorem in the previous problem. Go back and read the statement and make sure you understand it. You are now going to practice **applying it**.

   Prove that if f and g are two continuous functions such that $f(a) = g(a)$ for every $a \in \mathbb{Q}$ then actually $f(x) = g(x)$ for all $x \in \mathbb{R}$.

   (Hint: Consider the function $h(x) = f(x) - g(x)$. What do you know about $h(x)$? Use part a and this will be short.)

   > This is an important problem to understand. In this problem you had to **invent** the function $h(x)$ so that you could apply the theorem you proved in the earlier part. This will come up again in later homeworks.

16 Homework Due Tuesday October 24th at 5pm

1. Using the δ and ϵ definition of continuity, prove that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x - 6$ is continuous at $x = 7$.

2. Using the δ and ϵ definition of continuity, prove that $g : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 + 2$ is continuous at $x = -1$.

3. Choose one of the following and write a epsilon-delta proof that it is continuous at the given point:

$$f(x) = x^3, \quad at \quad x_0 = 1$$

$$g(x) = \sqrt{x}, \quad at \quad x_0 = 0$$

$$h(x) = \frac{1}{x^2}, \quad at \quad x_0 = 1.$$

   Friday's quiz will have a ϵ, δ proof on it.

4. In class we proved that every continuous function on a closed interval achieves a **maximum.** The analogous result is true for **minimum.** One way to do that would be to modify the big proof from class. (That's a lot of work). In this problem you'll prove this in a much shorter and easier way.

   (a) Prove the following general statement (note you don't need continuity or really anything here):

   "If $g : [a, b] \to \mathbb{R}$ achieves a maximum at x_0 then the function $-g$ achieves a minimum at x_0."

   (b) Now write a short but careful proof that "If $f : [a, b] \to \mathbb{R}$ is continuous then f achieves a minimum on $[a, b]$." Your proof should **use** part a) and it should use the Extreme Value Theorem. Be specific about what your function g is.

5. Let f and g be real valued functions. This problem is about understanding what taking the 'minimum' of two functions does.

   a. As a warmup, go on Desmos and type $\min(x + 1, x^2 - 1)$. Do you understand what you are seeing? Graph of $y = x + 1$ and $y = x^2 - 1$ and make sure you see the connection.

   b. The way to define the minimum function is to just look at which value is smaller:

$$\min(f, g)(x) = \begin{cases} f(x) & \text{if } f(x) \le g(x) \\ g(x) & \text{if } g(x) < f(x) \end{cases}$$

   I want you to **check** that:

$$\min(f, g)(x) = \frac{1}{2}(f + g) - \frac{1}{2}|f - g|.$$

   (Note this proof should be short. You should take two cases. Start your proof like this: "Let $p \in \mathbb{R}$. Case 1: $f(p) \le g(p)$. Now $\min(f, g)(p)$ =???? and [now check the right hand side and show they match. Then continue with case 2].

c. By using the Main Theorems on Continuity (from Day 20), prove that $\min(f,g)$ is continuous if f,g are continuous. Be sure to clearly state what theorems you are using and say why they apply. **What you have just done - is rewrite a "weird" function using "min" in terms of nice functions like sums, multiples, and absolute values. This is a very useful skill in math.**

17 Homework Due Friday October 27th at 5pm

1. Give **TWO** proofs that the function

$$g(x) = \begin{cases} \sin(\frac{1}{x^2}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is discontinuous at $x = 0$; one using the $\delta - \epsilon$ condition (you must!) and one using the sequential definition (you must!).

> Note: Remember, showing something is NOT continuous using sequences is pretty easy. (You remember this strategy from last week's HW / Quiz, right?)
>
> When using the δ, ϵ condition it's a little harder, but here's a hint. To show something is NOT continuous, you just have to show that there's some ϵ for which you can't find a δ. Let $\epsilon = 0.1$ and show that there's no δ that works.
>
> π should be involved.

2. Reread the proof that a continuous function achieves its min and max on the interval $[a,b]$. Where does the proof break down if the closed interval is replaced by the open interval (a,b)?

3. A function $f : D \to \mathbb{R}$ is called **Lipschitz** if there is a constant $K > 0$ such that for all $x, y \in D$, $|f(x) - f(y)| \leq K|x - y|$. Note that if $x = y$ then this just says $0 \leq 0$ which is always true. We can restate this as:

$$f \text{ is Lipschitz if } \quad \exists K \ : \ \forall x, y \in D \text{ with } x \neq y \quad \frac{|f(x) - f(y)|}{|x - y|} \leq K.$$

   (a) Verify that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 0$ is Lipschitz. (What is a value of K that works)?

   (b) Rephrase the Lipschitz condition in terms of slopes of secant lines. Draw a picture with a function $f(x)$ and points x and y and say what this means about the slope of the secant line between the points on the graph.

   (c) Think about why the function $f : (0,1) \to \mathbb{R}$ given by $f(x) = \sqrt{x}$ is NOT Lipschitz. I'll ask you to prove this next week.

   (d) Is the function $f : [0,2] \to \mathbb{R}$ given by $f(x) = x^2$ Lipschitz? (This is easiest to answer by just thinking about the slopes of secant lines. You don't need to be super rigorous here, just tell me what your K is and why. Feel free to appeal to geometry / calculus. This problem is just about stretching our brains.)

   (e) Is the function $f : (0,\infty) \to \mathbb{R}$ given by $f(x) = x^2$ Lipschitz?

Optional (Good Proof Portfolio Project) Recall that each rational number can be written uniquely as $\frac{p}{q}$ where p and q have no common factors and $q > 0$. For each $x \in \mathbb{Q}$, define f by $f(x) = \frac{1}{q}$ (where q is this unique denominator) for each $x \in \mathbb{R} \setminus \mathbb{Q}$ set $f(x) = 0$. Prove that $f(x)$ is

(a) discontinuous at every $a \in \mathbb{Q}$.

(b) continuous at every $a \in \mathbb{R} \setminus \mathbb{Q}$.

18 Homework Due Tuesday October 31st at 5pm

1. Prove that the function $f : (0, \infty) \to \mathbb{R}$ given by $f(x) = 1/x^2$ is continuous at $x = 1$.

   Hint: A few people chose this for hw last week, but no one got it completely correct. As a hint, you will not want to let $\delta < 1$, because this will only ensure that $0 < x < 1$ and then $1/x^2$ could be 10000000000000000000 (or larger). Instead, try using $\delta < 0.5$, so then you'll have that $0.5 < x < 1.5$, and now what's the biggest that $1/x^2$ can be? Be careful here and think about Friday's quiz.

2. Using the Intermediate value theorem, solve the following problems:

   I want you to be very specific with these. You must say:

   - What is your function f? Why is f continuous?
   - What are your values of a, b, W.
   - And then when you use the IVT you get a number c. Make sure you give a conclusion.

   (a) Show that if f and g are continuous functions on the interval $[a, b]$ such that $f(a) < g(a)$, $g(b) < f(b)$ then there is a number $c \in (a, b)$ such that $f(c) = g(c)$. (Hint: draw the graphs, and maybe cook up a new function $h(x) = f(x) - g(x)$).

   (b) Let f be a continuous function from $[0, 1]$ to $[0, 1]$. Prove there is a point $c \in [0, 1]$ such that $f(c) = c$. That is, show that f has a fixed point. Hint: Apply the Intermediate Value Theorem to the function $g(x) = f(x) - x$.

   (c) Suppose that $f : [0, 2] \to \mathbb{R}$ is a continuous function and that $f(0) = f(2)$. Prove that there is a some number $c \in [0, 1]$ such that $f(c) = f(1 + c)$. Hint: you might want to use IVT and consider $g(x) = f(1 + x) - f(x)$. This is kinda cool. Draw your own crazy graph with $f(0) = f(2)$ and try to see where your c is. For example if your function measures temperature in minutes, this is says that assuming $f(0) = f(2)$ there's always some moment c where the temperature will be exactly the same 1 minute later!

3. Prove that $\sqrt{x}$ is not Lipschitz on $[0, \infty)$ (Check last week's homework for the definition of Lipschitz. This would be a great portfolio problem!)

4. Let $f(x) = \cos x$ With a calculator try taking $\cos(\cos(\cos(\cos(\cos(0)))))$ and $\cos(\cos(\cos(\cos(\cos(1)))))$ and maybe $\cos(\cos(\cos(\cos(\cos(235)))))$. It turns out that no matter what input you start with you will always converge to the same number. This is cool. We'll prove this later. Do you think this is cool?

   Note: These types of functions occur all the time. For instance, we sometimes have functions which are called contractions, which occur when f is Lipschitz and $0 < K < 1$. For instance, imagine a function such that the function shrinks points more closely together, so $f(x)$ and $f(y)$ are closer together than x and y.

   Yo doc, is this problem even asking us to do anything? Yes - it asks one question :)

By the end of this unit, we will have proved that

i) A **continuous** function on a **closed** interval is always uniformly continuous on that interval.

ii) A Lipschitz function is uniformly continuous on its domain.

iii) A continuous function $f(x)$ is uniformly continuous on an open interval if and only if it can be extended to the include the end points so that the extension is continuous. (Think about "filling in values for $f(a)$ and $f(b)$ that make the whole function continuous on the closed interval).

iv) A uniformly continuous function on any bounded interval (a, b) or $[a, b]$ must be bounded.

v) If f is differentiable (has a derivative) on its domain and this derivative is **bounded**, then f is uniformly continuous. This works even on infinite intervals. You'll use this below.

5. Which of the following functions are uniformly continuous on the indicated set. Justify your answers and use any theorems you wish. You should find yourself using the rules in the box above. **Aim to find the most simple reason you can - some of these will be really quick!**

- $A(x) = x^2$ on $[3, 4]$
- $B(x) = x^2$ on $(5, 8)$
- $C(x) = x^2$ on $\mathbb{R}$
- $D(x) = \frac{1}{x^3}$ on $[1, 8]$
- $E(x) = \frac{1}{x^3}$ on $[1, \infty)$
- $F(x) = \frac{1}{x^3}$ on $(0, 1)$
- $G(x) = \tan x$ on $[0, \pi/2)$
- $H(x) = x \sin x$ on $(0, 1)$

19 Homework Due Tuesday November 3rd at 5pm

O. Ross 19.1 (don't need to write all these down, but be familiar with the theorems that you will use.)

1. Ross 19.2a,b,c These are good practice with Delta / Epsilon proofs!

2. Prove that if $f : D \to \mathbb{R}$ is Lipschitz then f is uniformly continuous on D. (This proof should be very short).

3. Is the function $f(x) = x^2$ uniformly continuous on $(0, \infty)$? (No, it isn't) Prove your answer using the ϵ, δ definition. Hint: Show that if $\epsilon = 1$ then there is no δ that "works" for all points. Your proof can begin with "Let $\epsilon = 1$. I will show that no matter what δ is, I can find a point x_0 where δ doesn't work. This will mean that I can find a point x such that $|x - x_0| < \delta$ BUT $|f(x) - f(x_0)|$ is NOT less than ϵ.

4. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous at a and that $f(a) > 0$. Prove that there is some $\delta > 0$ such that if $|x - a| < \delta$ then $f(x) > 0$. (Hint: the ϵ, δ definition will help here.)

O (Optional, but good for quiz practice or proof portfolio.

Let $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Prove that f is continuous at 0, but that f is NOT continuous at any other point.

20 Homework Due Tuesday November 7th at 5pm

What's on the Quiz? The quiz this **Wednesday** will be about uniform continuity. I won't ask you to prove anything, but instead I will ask you to supply examples and reasons. I will just ask a smattering of questions like the front page of the uniform continuity packet (the one with all the examples). This will be another group quiz.

1. Let $g(x) = x^2$ and let f be:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

   (a) For which x is $f(x) \leq x$?

   (b) For which x is $f(x) \leq g(x)$?

2. In class we showed that if f is uniformly continuous on $(0,1)$ then if a_n is Cauchy, then so is $f(a_n)$. Find an example of a Cauchy sequence a_n and a continuous function f on $(0,1)$ such that $f(a_n)$ is not Cauchy. This shows we really do NEED uniform continuity, and that continuity is not enough.

3. Let $f : [a,b] \to \mathbb{R}$ be a continuous function which has the property that for each $x \in [a,b]$ there is a $z \in [a,b]$ such that $|f(z)| \leq \frac{1}{2}|f(x)|$. You will show that such a function MUST equal 0 at some point in $[a,b]$. Please write your solution carefully. Here are some hints:

   (a) Start by plugging in $x = a$. Draw a picture and label $f(a)$ on the y-axis. For the sake of a picture imagine that $f(a)$ is positive, maybe up around 8. Now what does the condition say about the function? Your answer should involve some other point z. Label this on your picture and put it its y-value. Make sure you draw this to scale. What do you notice?

   (b) What if you repeated this, now using $x = z$, what would you get? You should get a third point w whose y value satisfies $f(w) \leq \frac{1}{4}f(a)$.

   (c) Now rather than calling these points a, z, w why not build a sequence? Can you write a sentence or two defining a sequence $x_1 = a$, $x_2 = ??$, etc. with some nice properties? What are the outputs converging to? Why? Can you use continuity.

   (d) Your proof should completely prove that Now using continuity, prove that there is a point $c \in [a,b]$ such that $f(c) = 0$.

21 Homework Due Friday November 10th at 5pm

1. In class we did something totally deranged - and talked about derangements. We saw that the probability that a permutation is a derangement converges to $1/e$ as the number of objects grows. E.g. on 4 letters ABCD we saw that there were 9 derangements. Let's define $T(n)$ to be the number of derangements on n letters. In class we worked out some of these numbers:

n	1	2	3	4	5	6	7
$T(n)$	0	1	2	9	44	265	1854

   And we found a formula for this number (based on noticing a pattern).

In class we saw (based on a pattern) that:

- $T(n) = n!\left(1 - 1 + 1/2 - 1/3! + 1/4! - 1/5! + 1/6! - \ldots + (-1)^n \frac{1}{n!}\right)$
- $1/e = 1 - 1 + 1/2 - 1/3! + 1/4! - 1/5! + 1/6! - \ldots + (-1)^n \frac{1}{n!} + \cdots$
- And thus the probability that a permutation is a derangement is given by

$$\frac{T(n)}{n!} = \left(1 - 1 + 1/2 - 1/3! + 1/4! - 1/5! + 1/6! - \ldots + (-1)^n \frac{1}{n!}\right) \approx \frac{1}{e}.$$

(a) Confirm with a calculator that this formula for $T(n)$ works for $n = 5$ and $n = 6$ (matching the table above).

(b) Now define this weird function $W(n) = [n! \cdot (1/e)]$ where $[x]$ means "round x up or down to the nearest integer." With a calculator, calculate $W(5)$ and $W(6)$. What do you notice?

(c) You should have noticed that $W(n) = T(n)$, which is a much simpler way to calculate $T(n)$. Prove that this is true. That is, prove that for all integers n,

$$n!\left(1 - 1 + 1/2 - 1/3! + 1/4! - 1/5! + 1/6! - \ldots + (-1)^n \frac{1}{n!}\right) = [n! \cdot (1/e)].$$

Hint: This problem is legitimately challenging and requires thinking about series, sequences and approximations. You might find the following hint helpful:

$$1/e = 1 - 1 + 1/2 - 1/3! + 1/4! - 1/5! + 1/6! - \ldots + (-1)^n \frac{1}{n!} + \cdots$$

that is:

$$1/e = \frac{T(n)}{n!} + \left(\frac{(-1)^{n+1}}{(n+1)!} + \frac{(-1)^{n+2}}{(n+2)!} + \cdots\right)$$

(Optional) Some enrichment - Take a look at the wikipedia article

`https://en.wikipedia.org/wiki/Weierstrass_function`

which describes the Weierstrass function which is an example of a function that is continuous but that is not differentiable anywhere. This was a very important part in the history of mathematics. It shows that continuous functions can be "strange".*

On the next page are some optional problems to get practice calculating limits. I recommend you at least look at them to see what things might show up in the future.

*Related: `https://tinyurl.com/StrangeContinuity`

(Optional) Compute the following limits. (You may just use the basic algebra rules - this is just review from Calculus)

- $\lim_{x\to 2} \frac{x^2+x-2}{x^2-1}$.
- $\lim_{x\to \infty} \frac{x^2+x-2}{x^2-1}$.
- $\lim_{x\to 2} \left(\frac{x^2-4}{x-2}\right)^{3/2}$.

(Answers: The limits are $4/3, 1, 8$.) (Optional) For extra practice with absolute value try these problems. For each of these I want you to simplify the absolute value like this: "Hmm, I am simplifying $|x-4|$ and I know that $x > 4$. this mean the thing inside the absolute value is already positive so I don't need to change anything, so I'll write:

$$|x-4| = x-4$$

and continue simplifying."

Or

"Hmm, I am simplifying $|x-4|$ and I know that $x < 4$. this mean the thing inside the absolute value is negative so I need to switch the sign. I'll write:

$$|x-4| = 4-x$$

and continue simplifying."

- $\lim_{x\to 2^+} \frac{|x-2|}{x^2-4}$
- $\lim_{x\to 2^-} \frac{|x-2|}{x^2-4}$
- $\lim_{x\to 3^-} \frac{|3-x|}{x^2-9}$
- $\lim_{x\to 3^-} \frac{x-3}{|x^2-9|}$

Check your answers with Desmos.

22 Homework Due Tuesday November 14th at 5pm

In Calculus, you learned about derivatives, limits and integrals. We'll be covering this material in more depth this semester. For this homework I want you to explore the **definition** of the derivative. We say that a function $f(x)$ is differentiable at a point $x = a$ if the following limit exists:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

OR

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

1. (Optional Warmup Problems):
   - Using the definition of derivative (as a limit above), prove that the derivative of $1/x$ is $-1/x^2$. I.e. show that $f'(a) = -1/a^2$.
   - Using the definition of the derivative, compute the derivative of $g(x) = x^2 + 3x$.
2. (Optional) Using the chain rule and any other rules you know from calculus, calculate the derivative of the functions below.
   - $\tan\left(\frac{x}{x^2+1}\right)$
   - $x^2 \sin(1/x)$. (provided $x \neq 0$) your answer to this part will be useful for problem 4.
3. (This problem doesn't use any derivatives - this problem will be an EXCELLENT problem for your proof portfolio. Please come and talk with me about it. It's ok if you don't completely solve it by this week.) We say that a function $f : \mathbb{R} \to \mathbb{R}$ is a contraction mapping if there is an r, $0 \leq r < 1$ such that for all $x, y \in \mathbb{R}$,

   $$|f(x) - f(y)| \leq r|x - y|.$$

   In other words the function is Lipschitz with constant less than 1. Recall that this is saying that all slopes of secant lines between any two points on the graph of $f(x)$ have slope less than r. Show that if f is a contraction mapping then it has a fixed point, that is, the equation $f(x) = x$ has a solution! In fact, show that if you start with some point in the domain, (to get started, let $x_0 = 0$ and just iterate your function: $x_1 = f(x_0)$, $x_2 = f(x_1), \ldots$, then x_n is a Cauchy sequence and (thus convergent to some point x). Show that $f(x) = x$.

   For hints: click this number: 28

   If you want to see this in action, find a fixed point for $\cos(x)$. I just tried this with google, because I don't have a calculator handy, and I typed:

   $$\cos(\cos(\cos(\cos(\cos(\cos(\cos(\cos(\cos(\cos(\cos(\cos(\cos(\cos(\cos(\cos(\cos(0)))))))))))))))))$$

   and got .739, and then sure enough, cos(.739) = .739 (well, roughly)

   (Note: strictly speaking I'm looking at $\cos(x) : [0,1] \to [0,1]$, where it is true that the derivative is always strictly less than 1.)

23 Homework Due Friday November 17th at 5pm

1. The function given by

$$f(x) = \begin{cases} x^2 + 2 & x > 0 \\ 0 & x \le 0 \end{cases}.$$

   Is not differentiable at 0.

   (a) Prove this by showing that f is not continuous at zero and by citing the appropriate theorem we learned in class.

   (b) Prove this by using the definition of $f'(0)$ and showing that the limit does not exist.

2. On earlier homework you showed before that the function $f : \mathbb{R} \to \mathbb{R}$ given below **IS** continuous at 0.

$$f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

   Show that it is not differentiable at 0. (Hint: Use the $f(x) - f(a)$ definition in the box above, simplify and explain why the limit doesn't exist. Your explanations can be brief and refer to similar previous hwk).

3. Consider now the function $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

   Show that this function is indeed differentiable at a for all real numbers a. Write down a formula for the derivative. (Your formula will have cases, you'll have to compute the derivative as a limit at 0 and you can use the product / chain rule for all other points). Explain why $g'(x)$ is not continuous at 0.

You have just seen two cool facts:

- Some functions are **continuous** but are not differentiable.
- Some functions are **differentiable** but their derivatives are **not continuous.**

4. (Not to turn in) Some enrichment - Take a look at the wikipedia article

   `https://en.wikipedia.org/wiki/Weierstrass_function`

   which describes the Weierstrass function which is an example of a function that is continuous but that is not differentiable anywhere. This was a very important part in the history of mathematics. It shows that continuous functions can be "strange".[†]w

[†]Related: `https://tinyurl.com/StrangeContinuity`

24 Homework Due Tuesday November 21st at 5pm

This HW is all about using the Mean Value Theorem.

1. Let f be a differentiable function on $\mathbb{R}$. Prove that if there is an r such that $|f'(x)| \le r$ for all $x \in \mathbb{R}$ then
$$|f(x) - f(y)| \le r|x-y|, \quad \text{for all } x, y \in \mathbb{R}.$$
(Hint: Begin your proof by saying "Let $x, y \in \mathbb{R}$. By the mean value theorem we know that... "

2. Using the idea in Problem 1, prove that if x and y are real numbers, then $|\sin x - \sin y| \le |x-y|$. You may use that the derivative of $\sin x$ is $\cos x$ and that sine and cosine are bounded between -1 and 1.

3. Let $r > 0$ and let $r \le x \le y$. Prove that $\ln y - \ln x \le \frac{y-x}{r}$.

4. Is the function below differentiable at 0?

$$f(x) = \begin{cases} \frac{1}{2}x & \text{if } x \in \mathbb{Q} \\ x & \text{if } x \notin \mathbb{Q}. \end{cases}$$

25 and 26 and 27 - Last ones!

Instructions:

We have three more homework assignments due. I want you to choose your own adventure in terms of work. For each of these assignments you should solve **at least two** of the problems in this document.

All of these problems would be suitable for our final exam, so you might want to try and solve them all.

In addition you should be working on your proof portfolio document. If you come by my office hours you can show me **as many proofs as you like** and I can offer feedback before you turn in your proof portfolio. I will be holding lots of extra office hours the last few weeks of school.

On this problem you may use the following theorems that we won't get to prove this semester. These are theorems you have seen before in Calculus:

Some Definitions

1. If f is integrable on $[a,b]$ then we define $\int_b^a f(x)dx = -\int_a^b f(x)dx$.
2. $\int_a^a f(x)dx$ is defined to be zero.
3. Remember that if $a < b < c$ and if f is integrable on $[a,b]$ then you can break up the integral from a to c as a sum of integrals from a to b and then b to c.

Theorem (u-substitution)

Suppose that g is a function whose derivative g' is continuous on $[a,b]$ and suppose that f is a function that is continuous on $g([a,b])$. Then

$$\int_a^b f(g(x))g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)\ du.$$

Don't overthink this one - this is the u-substitution you know and ♡.

1. Use u-substitution to rewrite the following x integrals are u integrals. Type both the original integral and the converted one into Desmos to make sure you get the same numerical value.

   I am NOT asking you to find antiderivatives - in fact I actually don't want you to find any antiderivatives on this homework. I want you to rewrite these integrals and practice changing the bounds, and check that the numbers you get on Desmos are equal. The point it to develop your skills of carefully making substitutions. Taylor Swift will require your assistance with these carefully honed skills later.

   (a) $\int_1^5 (2x+1)^8\ dx, (u = 2x+1)$

   (b) $\int_0^\pi \cos(x/4)\ dx, (u = x/4)$

   (c) $\int_1^5 \frac{1}{6x}\ dx, (u = 6x)$

   (d) $\int_1^{1/7} \frac{1}{x}\ dx, (u = 7x)$

2. (All I want for Christmas is u)

Suppose that Mariah Carey defines the function

$$M(x) = \int_1^x f(t)\ dt$$

where f is a function that is continuous on $(0, \infty)$.

(a) Explain carefully why $M(4)$ is defined. I want to see sentences explaining why an integral exists.

(b) Explain carefully why $M(1/3)$ is defined. I want you to use the theorems we have learned in class, and write $M(1/3)$ in terms of an integral $\int_{blah}^{blah}$ where the upper numbers is bigger than the lower one.

(c) Beyoncé wants to calculate the number $\int_{1/4}^{6} f(t)\ dt$. Help Beyoncé write this number down in terms of $M(x)$. Your answer should have no NO integrals in it - I want you to do this in terms of $M(\text{stuff})$.

(d) Rewrite the integral below completely in terms of some $M(blah)s$. You will need to use u-substitution.

$$\int_4^5 f(5t)\ dt.$$

3. Taylor Swift, in rare form, decides she wants to see what it feels like to take an L, so she defines the function:

$$L(x) = \int_1^x \frac{1}{t}\ dt.$$

On this problem you are NOT allowed to use logarithms or find any antiderivatives. Instead, using u-substitution, I want you to study the following things:

(a) Suppose that $a > 1$. Help Taylor write $L(1/a)$ in terms of $L(a)$ by using the substitution like in Problem 1d. I'll get u started: Let $u =$ ____________.

$$L(1/a) = \int_1^{1/a} \frac{1}{x}\ dx =$$

I want you to be very careful here. This is a type of question I might ask on the final exam.

You should have found a very nice property about Taylor's rare L. Now let's prove some more:

(b) Suppose that $a, b > 1$. Prove that $L(ab) = L(a) + L(b)$. I recommend you find the right substitution for the left hand side and then simplify it down until you get the right hand side.

(c) Using the previous parts and the fact that $a/b = a(1/b)$, conclude that $L(a/b) = L(a) - L(b)$.

(d) What is $L(1)$?

(e) Surprise - Taylor never takes an L, in fact, she has just discovered the **natural log function.** This is how we **can define** the natural log of a number in terms of area:

$$\ln(x) = \int_1^x \frac{1}{t}\ dt.$$

Go on Desmos.com and type

$$\int_1^a \frac{1}{t} dt$$

and move the a slider around until the value is 1. You should see that this number is e. With further effort, we could show that this number is the same as $1 + 1 + \frac{1}{2} + \frac{1}{3!} + \cdots$

4. Let $f : [0,2] \to \mathbb{R}$ be defined by $f(x) = x^2 - x$ and let $P = \{0, 1, \frac{3}{2}, \}$. Compute $U(f,P)$ and $L(f,P)$.

5. One of the best ways to show that a function is integrable is to use the "Integrals Analytically Theorem." This means you start with "Let $\epsilon > 0$ and then you find a partition so that $U - L$ is $< \epsilon$. Use this method to show that the following function is integrable on $[0,3]$:

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1) \\ 2 & \text{if } x \in [1,2) \\ 3 & \text{if } x \in [2,3) \end{cases}$$

6. Do the same for the function

$$g(x) = \begin{cases} 1 & \text{if } x \in [0,1) \\ 5 & \text{if } x = 1 \\ 2 & \text{if } x \in (1,3) \end{cases}$$

   Your partition should be not too complicated, but it should be specific.

7. Consider the function: $f : [0,4] \to \mathbb{R}$:

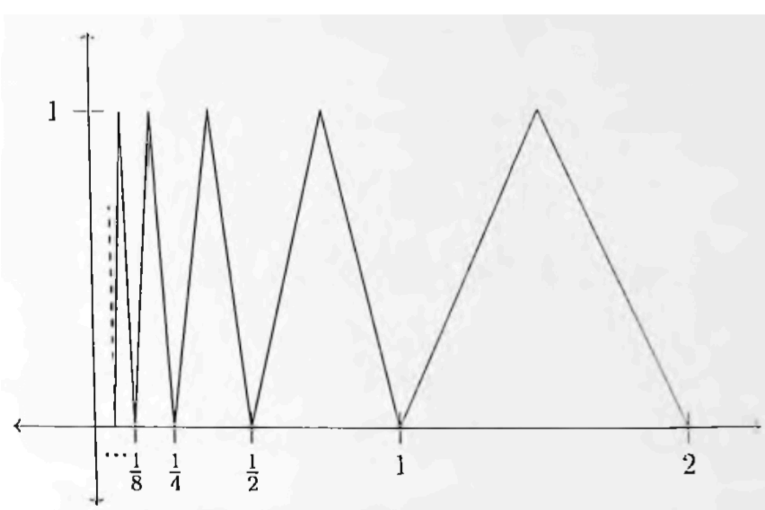
$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}.$$

   (a) Let P be a partition of $[0,4]$. Compute $L(g,P)$.

   (b) Find
$$\inf\{U(g,P) \ : \ P \text{ is a partition of } [0,4]\}.$$
   (Don't worry about the proof of this - but I want you to think about how you know this is the Upper sum.

8. Give an example of numbers a, b and of two integrable functions $f, g : [a,b] \to \mathbb{R}$ such that for **any** partition P:
$$U(f+g,P) < U(f,P) + U(g,P).$$
   Give a careful explanation of your answer.

9. Suppose that f is the function depicted below, where the pattern continues. Is f integrable on $[0,2]$? Prove your answer. Hint: the answer is yes, and you actually don't have to work very hard if you use the "Integrals Analytically Theorem" carefully and cite another theorem.



10. For practice, write your own proof why the function:

$$f(x) = \begin{cases} 1 & x \text{ is rational} \\ 0 & x \text{ is irrational} \end{cases}$$

is NOT integrable on the interval $[0,1]$. Remember, just because it's not continuous doesn't mean it isn't integrable, so you'll have to work out the upper and lower sums.

11. Give an example of a function f such that $|f|$ is integrable on $[0,1]$ but f is not integrable on $[0,1]$. Hint: How could you modify the function in the previous problem?

12. Prove that

$$1 \leq \int_{-1}^{1} \frac{1}{1+x^{2n}} dx \leq 2$$

for all $n \in \mathbb{N}$. Hint: what is the max and min of this function on the interval? What Theorem can you use to help?

28 Hints for Contraction Mapping Problem

Here are some ways that we can think about Contraction mappings:

- What is f? Is it $\cos x$?
  - Answer: No - $\cos x$ was just an example. The problem says that f is a contraction mapping.
- What's a contraction mapping?
  - Answer: In problem 1, a contraction mapping was any function f so that there was some number $0 < r < 1$ such that
    $$|f(x) - f(y)| \leq r|x - y|$$
    for all x, y.
- Haven't we seen something like that before?
  - Answer: Yes you have, this is the Lipschitz condition about which you've proven a very strong property.
- Does the old Homework help with this problem?
  - Answer: Well Lipschitz functions are ________________________. That seems like a hecka good property that we proved lotsa things about...

Ok so this problem is about contraction mappings - what are those? Well let's look at

$$|f(x) - f(y)| \leq r|x - y|.$$

If we think of x and y as our inputs, this says, if I have two inputs that are some distance apart, then the distance between their outputs is at most r times that distance. E.g. if $|x - y| = 10$ and $r = 1/3$ then $|f(x) - f(y)| \leq 10/3$. Wow, those outputs are pretty close together - compared to the inputs. Imagine that you are x and your friend who lives 10 miles away is y. After you apply this function f, now you and your friend are only $10/3 = 3.33$ miles apart. Whoa this isn't that different from last week's homework where we had outputs that we getting smaller and smaller by $1/2$ at a time. Neato.

Some questions to think about

- Let $f(x) = x^2 + 1$
- Let $f(x) = 3x + 5$

Find two inputs x and y that are 10 apart and feed them into your function $f(x)$ and see how far apart the outputs are? Is it more or less than 10? Since e.g. the pair $x = 0, y = 10$ goes to points that are more than 10 apart, these functions are NOT contraction mappings.

After all, what are are we trying to prove anyway? That's a good question - we've been so busy trying to figure out what our function was doing, that we forgot to look at the goal. We are asked to "show that f has a fixed point." which means that $f(x) = x$ has a solution. Let's put that in a box:

Show that IF f is a contraction mapping THEN f has as fixed point.

Hmm, this will mean that the graph of $y = f(x)$ should intersect the graph of the line $y = x$.

- Hmm should that always happen for any function?
  - No, I guess not, I can draw graphs that don't hit that line. For instance, x^2+1 won't hit the line $y = x$ (aha, that's good, since I worked out that x^2+1 wasn't a contraction mapping.)
- **Wait a minute, but $3x+5$ DOES intersect the line $y = x$ and $3x+5$ isn't a contract mapping either!**
  - That's true - very good! This means that that converse is false.

f is a contraction mapping $\implies$ f has as fixed point. TRUE

f has a fixed point $\implies$ f is a contraction mapping. FALSE

Ok, how can I solve something like this?

Admittedly this is a hard problem, and so I've given a hint - in fact a way to find that fixed point exactly! I say - start with 0, and then just keep plugging it into your function and keep track of what happens:

$$x_0 = 0$$

$$x_1 = f(0)$$
$$x_2 = f(x_1)$$
$$x_3 = f(x_2)$$

etc. I claim that this sequence x_n is a Cauchy sequence, which means it converges to some limit a. This is just like you did with $\cos\cos\cos\cos\cos()$ etc.

TODO: Prove that x_n is Cauchy.

TODO: Figure out what this limit a has to do with the problem.

Let's first figure out what the limit a has to do with the problem. Well in the example with $\cos(x)$ it looks like the iterates are getting closer and closer to the fixed point. So maybe it's true that $f(a) = a$. Let's make that our goal:

Goal: Show that x_n is Cauchy. Call the limit a. Then prove that $f(a) = a$.

Note that this will solve the problem if we are successful. There's a chance that maybe a isn't a fixed point, but this still seems like a promising approach, so let's do it!

How to show $f(a) = a$: What do we know about a? We just know that it is the limit of x_n. So we know that $x_n \to a$.

- Wouldn't it be great if we knew that $f(x_n) \to f(a)$? (Answer: yes, that would be great!)
- What property of f would guarantee this? (Answer: When f is ______________________)
- Did a previous homework problem tell us anything about this?

Ok so $f(x_n) \to f(a)$. **But how do I show that $f(a) = a$??** Well, maybe you can show that $f(x_n)$ also converges to a. Since it converges to $f(a)$ and a, then those must be the same number! **Well how can I do that?** Why don't you look at the definition for $f(x_n)$? Wasn't a the limit of the x_n's? What's the relationship between x_n and $f(x_n)$? (This is the part of the proof where there's no real way to explain it other than just thinking through what the sequence and limits are doing.)

How to show x_n is Cauchy

The definition of Cauchy is that $|x_n - x_m|$ is small when n and m are sufficiently big. Looking back at Homework 13.3 might be helpful.

Let's start by just looking at $|x_3 - x_2|$. Can we simplify this? Well $x_3 = f(x_2)$ and $x_2 = f(x_1)$. So this means that

$$|x_3 - x_2| = |f(x_2) - f(x_1)|.$$

Now can we make this $\leq$ something? Well the contraction mapping property says this should be at most $r|x_2 - x_1|$. This means that

$$|x_3 - x_2| = |f(x_2) - f(x_1)| \leq r|x_2 - x_1|$$

Now let's repeat! what's $|x_2 - x_1|$? Well $x_2 = f(x_1) = \ldots$. You should get:

$$|x_3 - x_2| = |f(x_2) - f(x_1)| \leq r|x_2 - x_1| = r|f(x_1) - f(x_0)| \leq r^2|x_1 - x_0| = r^2|f(0)|.$$

TODO: Do the same with $|x_4 - x_3|$ and look for a pattern

TODO: What does the pattern say that $|x_{n+1} - x_n|$ would be less than?

If you want at this point, you can just cite Homework 7.4c and say that this is sufficient to prove that x_n is Cauchy, or else you can mimic the proof of that homework problem (see solutions)).

And we're done!

- My hope in typing these notes was to show you that although this problem has a lot of steps, it combines ideas that are central to the course. How do we prove something's Cauchy? We examine the difference between terms. How do we do that? We use inequalities and the given terms. Cauchy, convergent, limit, etc.
- This problem was challenging, but I encourage you to write up your solutions carefully and to think through each of the steps. This is a great chance to show all that you know about analysis!